Magnetic Effect of Current

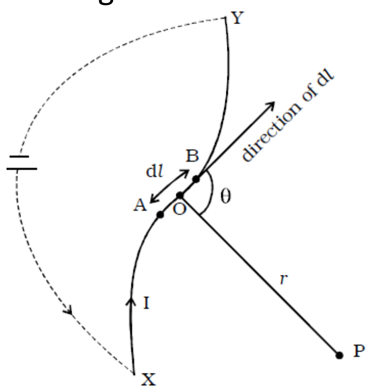
SECTION I

The magnetic field:

Magnetic field is the region around the moving charge in which magnetic force is experienced by the magnetic substances.
Magnetic field is a vector quantity and also known as magnetic induction vector. It is represented by **B**
Lines of magnetic induction may be drawn in the same way as lines of electric field.
The number of lines per unit area crossing a small area perpendicular to the direction of the induction being numerically equal to **B**.
The number of lines of **B** crossing a given area is referred as the magnetic flux linked with that area.
For this reason, **B** is also called magnetic flux density.
The unit of magnetic field is weber/m^2 and also known as tesla (T) in SI system

BIOT-SAVART LAW:

Biot and Savart conducted many experiments to determine the factors on which the magnetic field due to current in a conductor depends. The results of the experiments are summarized as Biot-Savart law. Let us consider a conductor XY carrying a current I refer figure



AB = dl is a small element of the conductor. P is a point at a distance r from the midpoint O of AB. According to Biot and Savart, the magnetic induction dB at P due to the element of length dl is
(i) directly proportional to the current (I)
(ii) directly proportional to the length of the element (dl)
(iii) directly proportional to the sine of the angle between dl and the line joining element dl and the point P (sin θ)
(iv) inversely proportional to the square of the distance of the point from the element ($1/r^2$)

$$dB \propto \frac{I\,dl\,\sin\theta}{r^2}$$

$$dB = K\frac{I\,dl\,\sin\theta}{r^2}$$

K is the constant of proportionality; its value is μ /4π.

Here μ is the permeability of the medium. Value of K for vacuum is 10^{-7} wb/amp m.

$$dB = \frac{\mu}{4\pi}\frac{I\,dl\,\sin\theta}{r^2}$$

$\mu = \mu_r \mu_o$ where μ_r is the relative permeability of the medium and μ_0 is the permeability of free space. $\mu_o = 4\pi \times 10–7$ henry/metre. For air $\mu_r = 1$.

So, in air

$$dB = \frac{\mu_0}{4\pi}\frac{I\,dl\,\sin\theta}{r^2}$$

In vector form,

$$dB = \frac{\mu_0}{4\pi}\frac{\overrightarrow{I\,dl} \times \vec{r}}{r^3}$$

or

$$dB = \frac{\mu_0}{4\pi}\frac{\overrightarrow{I\,dl} \times \hat{r}}{r^2}$$

The direction of dB is perpendicular to the plane containing current element Idl and r (i.e plane of the paper) and acts inwards.

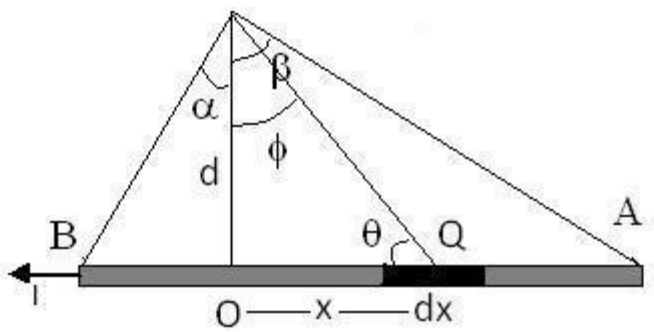
The unit of magnetic induction is tesla (or) weber $m^{-2.}$

Field due to a Straight current carrying wire

(i) When the wire is of finite length

Consider a straight wire segment carrying a current I and there is a point P at which magnetic field to be calculated as shown in figure.

This wire makes an angle of α and β at that point with normal OP. Consider an element of length dx at a distance y from O and distance of this element from point P is r and line joining P and Q makes an angle θ with the direction of current as shown in figure.



Using Biot-Savart Law magnetic field at point due to small current element is given by

$$dB = \frac{\mu_0 I}{4\pi}\left(\frac{dx\sin\theta}{r^2}\right)$$

As every element of the wire contributes to **B** in the same direction, we have

$$B = \frac{\mu_0 I}{4\pi}\int_A^B \left(\frac{dx\sin\theta}{r^2}\right)$$ .....eq(1)

From the triangle OPQ as shown in figure, we have

x = d tanϕ

Or dx = $d\sec^2\phi\, d\phi$

And in same triangle r = d sec ϕ and θ = (90° – ϕ)

Where ϕ is angle between line OP and PQ

Now equation (1) can be written as

$$B = \frac{\mu_0 I}{4\pi}\int_{-\beta}^{\alpha} \left(\frac{d\sec^2\varphi d\varphi \sin(90-\varphi)}{(d\sec\varphi)^2}\right)$$

$$B = \frac{\mu_0 I}{4\pi}\int_{-\beta}^{\alpha} \left(\frac{d\sec^2\varphi d\varphi \sin(90-\varphi)}{(d\sec\varphi)^2}\right)$$

$$B = \frac{\mu_0 I}{4\pi}\int_{-\beta}^{\alpha} \left(\frac{d\varphi\cos\varphi}{d}\right)$$

$$B = \frac{\mu_0 I}{4\pi d}\int_{-\beta}^{\alpha} (\cos\varphi d\varphi)$$

$$B = \frac{\mu_0 I}{4\pi d}[\sin\varphi]_{-\beta}^{\alpha}$$

$$B = \frac{\mu_0 I}{4\pi d}[\sin\alpha + \sin\beta]$$

Direction of **B**: the direction of magnetic field in determined by the cross product of the vector Idl with vector r

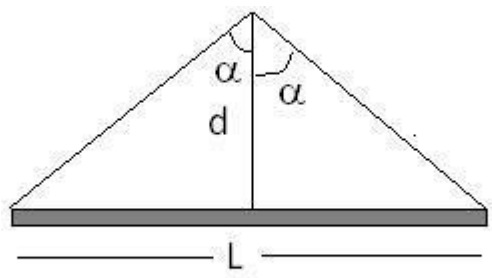
Therefore at point P the direction of the magnetic field due to the whole conductor will be perpendicular to the plane containing wire and point P or perpendicular to plane of paper and going into the plane

Case (I) when point P is on perpendicular bisector

In this case α = β using equation

$$B = \frac{\mu_0 I}{4\pi d}[\sin\alpha + \sin\beta]$$

$$B = \frac{\mu_0 I}{2\pi d}[\sin\alpha]$$



From figure

$$\sin\alpha = \frac{L}{\sqrt{L^2+4d^2}}$$

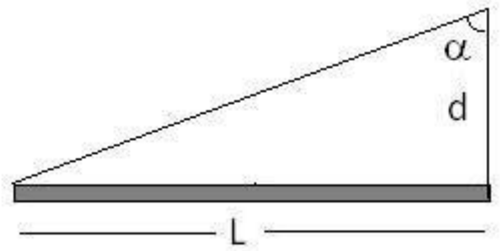
Case (II) when point P is at one end of conductor

In this case α =0 or β = 0

From equation

$$B = \frac{\mu_0 I}{4\pi d}\left[\sin\alpha + \sin\beta\right]$$

$$B = \frac{\mu_0 I}{4\pi d}\left[\sin\alpha\right]$$



From figure

$$\sin\alpha = \frac{L}{\sqrt{L^2+d^2}}$$

Case(III) When wire is of infinite length

In this case α = β = 90^{O}

From equation

$$B = \frac{\mu_0 I}{4\pi d}\left[\sin\alpha + \sin\beta\right]$$

$$B = \frac{\mu_0 I}{4\pi d}\left[\sin 90 + \sin 90\right]$$

$$B = \frac{\mu_0 I}{2\pi d}$$

Case(IV) When the point P lies along the length of wire (but not on it)

If the point is along the length of wire (but not on it) , then as vector dl and vector r will either be parallel or antiparallel i.e θ=0 or π,

From equation

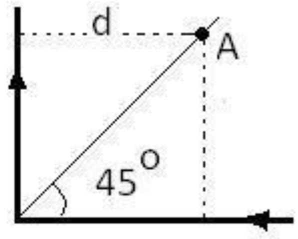
$$dB = \frac{\mu}{4\pi}\frac{\mathrm{I\ dl\ sin}\ \theta}{r^2}$$

$$dB = \frac{\mu}{4\pi}\frac{\mathrm{I\ dl\ sin}\ 0}{r^2}$$

$$dB = 0$$

Solved Problem

Q) A along straight conductor is bent at an angle of 90^{O} as shown in figure. Calculate the magnetic field induction at A



Solution:

For each portion α = 45 and β =90 .

From formula for magnetic field at a point

$$B = \frac{\mu_0 I}{4\pi d}\left[\sin\alpha + \sin\beta\right]$$

$$B = \frac{\mu_0 I}{4\pi d}\left[\sin 45 + \sin 90\right]$$

$$B = \frac{\mu_0 I}{4\pi d}\left[\frac{1}{\sqrt{2}} + 1\right]$$

$$B = \frac{\mu_0 I}{4\pi d}\left[\frac{\sqrt{2}+1}{\sqrt{2}}\right]$$

Each horizontal and vertical wires will produce same magnetic field at A and there directions are also same thus total field at A is

$$B = 2\times\frac{\mu_0 I}{4\pi d}\left[\frac{\sqrt{2}+1}{\sqrt{2}}\right]$$

$$B = \frac{\mu_0 I}{2\pi d}\left[\frac{\sqrt{2}+1}{\sqrt{2}}\right]$$

Q) A long straight wire carrying current produces a magnetic induction of 4×10^{-6}T at a point, 15 cm from the wire. Calculate the current through the wire.

Solution:

B =4×10^{-6}T , d = 15 cm = 0.15m

From formula

$$B = \frac{\mu_0 I}{2\pi d}$$

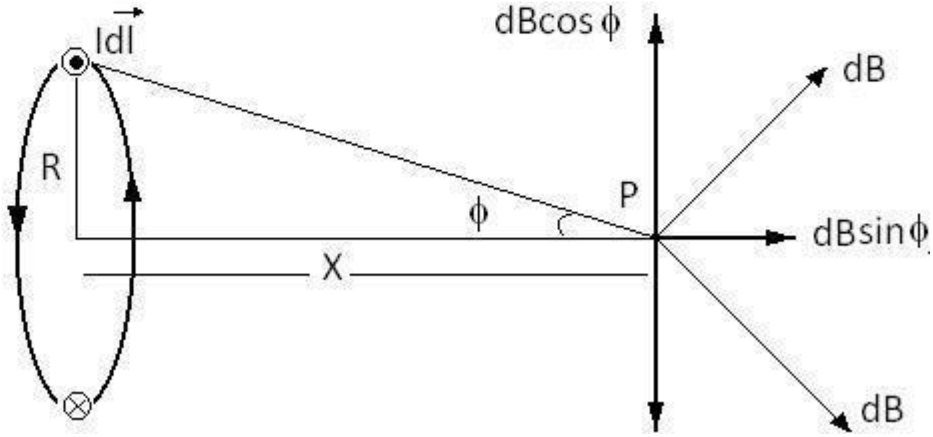
$$I = \frac{B(2\pi d)}{\mu_0}$$

$$I = \frac{4\times10^{-6}\times2\pi\times0.15}{4\pi\times10^{-7}}$$

$$I = 3A$$

Magnetic field at an axial point of a circular coil

Consider a circular loop of radius R and carrying a steady current I. We have to find out magnetic field at the axial point P, which is at distance x from the centre of the loop
X-axis is taken as along the axis of the ring.
Let the position vector of point P with respect to an element **dl** be **r** . The magnetic field dB at point due to current element I**dl** is in a direction perpendicular to the plane formed by **dl** and **r.**



Magnetic field at point P due to current element **Idl** is given by

$$\vec{dB} = \frac{\mu_0}{4\pi}\frac{i\vec{dl}\times\vec{r}}{r^3}$$

Since angle between Idl and r is 90^o we get

$$\vec{dB} = \frac{\mu_0}{4\pi}\frac{idl}{r^2}\hat{n}$$

Direction of magnetic field is perpendicular to plane containing dl and r as shown in figure
If ϕ is the angle between r and X, then from geometry of figure, component of **dB** along Y-axis will be dBcosϕ and dBsinϕ will be along X axis.
For all point on the circular coil there exists a diametrically opposite point such that magnetic field produced at point P cancels Y-component of first one thus resultant magnetic field at P is summation of X-component at P

$$B = \int dB \sin\varphi$$

$$B = \frac{\mu_0}{4\pi}\int \frac{Idl}{r^2}\sin\varphi$$

$$\sin\varphi = \frac{R}{r}$$

$$\sin\varphi = \frac{R}{\sqrt{R^2 + x^2}}$$

$$B = \frac{\mu_0}{4\pi}\int \frac{Idl}{r^2}\frac{R}{\sqrt{R^2 + x^2}}$$

$$B = \frac{\mu_0}{4\pi}\int \frac{Idl}{R^2 + x^2}\frac{R}{\sqrt{R^2 + x^2}}$$

$$B = \frac{\mu_0}{4\pi}\int \frac{IRdl}{\left(R^2 + x^2\right)^{3/2}}$$

$$B = \frac{\mu_0}{4\pi}\frac{IR}{\left(R^2 + x^2\right)^{3/2}}\int_0^{2\pi R} dl$$

$$B = \frac{\mu_0}{4\pi}\frac{IR}{\left(R^2 + x^2\right)^{3/2}}(2\pi R)$$

$$B = \frac{\mu_0}{2}\frac{IR^2}{\left(R^2 + x^2\right)^{3/2}}$$

Case (I) If the coil has N turns then

$$B = \frac{\mu_0}{2}\frac{NIR^2}{\left(R^2 + x^2\right)^{3/2}}$$

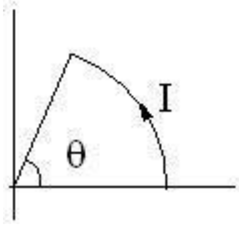
Case (II) Field at the centre of ring

In above equation x = 0

$$B = \frac{\mu_0}{2}\frac{NIR^2}{R^3}$$

$$B = \frac{\mu_0 NI}{2R}$$

Case (III) Magnetic field at the centre of a current arc



Form the equation for magnetic field at centre of coil. N is number of turns

2π = 1 turn

∴ θ = θ/2π turn s replacing value of N we get

$$B = \frac{\mu_0 I}{2R}\frac{\theta}{2\pi}$$

If l is the length of arc then l = θR above equation becomes

$$B = \frac{\mu_0 I}{2R}\frac{1}{2\pi}\frac{l}{R}$$

$$B = \frac{\mu_0 I l}{4\pi R^2}$$

Direction of magnetic field **B** for circular loop

Direction of magnetic field at a point the axis of a circular coil is along the axis and its orientation can be obtained using the right-hand thumb rule. If the fingers curled along the current then stretched thumb shows direction of magnetic field.

Magnetic field will be out of the page for anticlockwise current while into the page for clockwise current

Solved Problem

Q) A circular coil of 200 turns and of radius 20 cm carries a current of 5A. Calculate the magnetic induction at a point along its axis, at a distance three times the radius of the coil from its centre

Solution:

N=200, R = 0.2 m, I =5A , x = 3R

From the formula

$$B = \frac{\mu_0}{2}\frac{NIR^2}{\left(R^2 + x^2\right)^{3/2}}$$

$$B = \frac{\mu_0}{2}\frac{NIR^2}{\left(R^2 + 9R^2\right)^{3/2}}$$

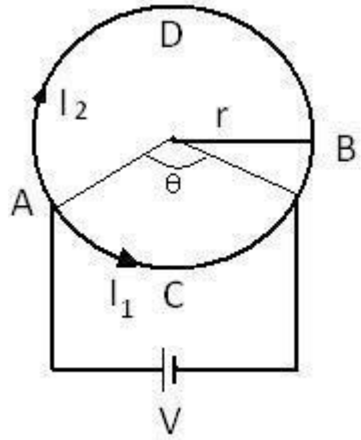
$$B = \frac{\mu_0}{2}\frac{NI}{10^{3/2} R}$$

$$B = \frac{\mu_0 NI}{20\sqrt{10}R}$$

$$B = \frac{4\pi\times 10^{-7}\times 200\times 5}{20\sqrt{10}\times 0.2}$$

$$B = 9.9\times 10^{-5} T$$

Q) A circular loop is prepared from a wire of uniform cross section. A battery is connected between any two points on its circumference. Show that the magnetic induction at the centre of the loop is zero



Solution:

Magnetic field at centre due to arc is given by

$$B_1 = \frac{\mu_0 I_1}{2R}\frac{\theta}{2\pi}$$

and

$$B_2 = \frac{\mu_0 I_2}{2R}\frac{(2\pi-\theta)}{2\pi}$$

Both magnetic fields B_1 and B_2 are in opposite directions

Let R_1 be the resistance of arc ACB and R_2 be the resistance of arc ADB since potential across both the resistance is same thus

$I_1R_1 = I_2R_2$ eq(1)

We also know that resistance ∝ length of wire

And lenth of wire ACB = rθ

Length of arc ADB = r(2π –θ)

Thus if ρ is resistance per unit length then

$R_1 = \rho r\theta$ and $R_2 = \rho r(2\pi - \theta)$

Thus equation (1) becomes

$I_1 \rho r\theta = I_2 \rho r(2\pi - \theta)$

$I_1 \theta = I_2 (2\pi - \theta)$ eq(2)

Now total magnetic field at centre

$$B = \frac{\mu_0 I_1}{2R}\frac{\theta}{2\pi} - \frac{\mu_0 I_2}{2R}\frac{(2\pi-\theta)}{2\pi}$$

from eq (3)

$$B = \frac{\mu_0 I_1}{2R}\frac{\theta}{2\pi} - \frac{\mu_0 I_1}{2R}\frac{\theta}{2\pi} = 0$$

Q) A charge Q is uniformly spread over a disc of radius R made from non conducting material. This disc is rotated about its geometrical axis with frequency f. Find the magnetic field produced at the centre of the disc.

Solution:

Suppose disc with radius R is divided into the concentric rings with various radii, Consider one such ring with radius r and thickness dr.

Total charge on disc = Q , charge per unit area $\rho = Q/\pi R^2$

∴ The charge on the ring with radius r = (area of the ring) (charge per unit area)

$q = (2\pi r dr)(Q/\pi R^2)$

If the ring is rotating with frequency f, then current produced I

$$I = \frac{Q}{\pi R^2}(2\pi r dr) f$$

This ring can be considered as circular loops carrying current I

Magnetic field at the centre due to this current will be

$$dB = \frac{\mu_0}{2r} I$$

$$dB = \frac{\mu_0}{2r}\frac{Q}{\pi R^2}(2\pi r dr) f$$

$$dB = \frac{\mu_0 Q f}{R^2}(dr)$$

∴ Magnetic field B produced at the centre due to the whole disc

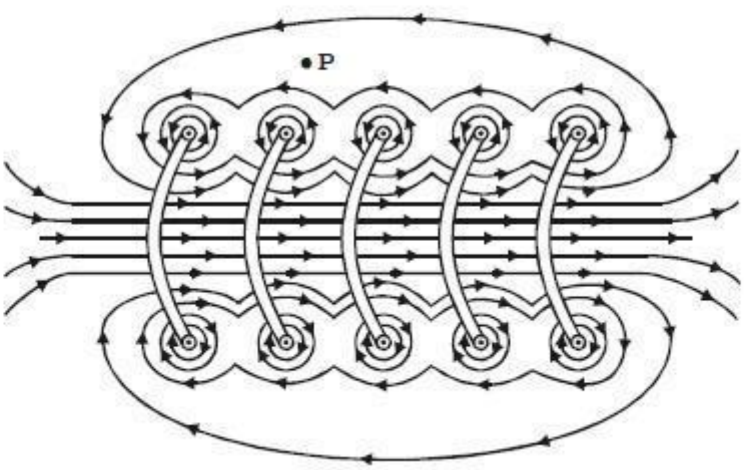
$$B = \int dB = \int_0^R \frac{\mu_0 Q f}{R^2} dr$$

$$B = \frac{\mu_0 Q f}{R}$$

Solenoid:

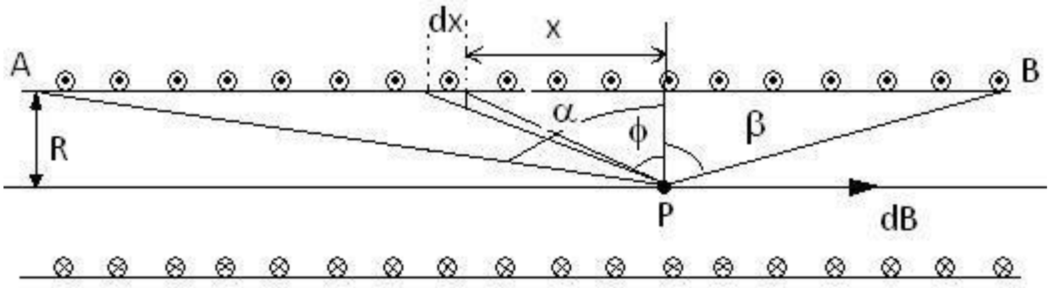
When two identical rings carrying current in same direction are placed closed to each other co-axially. It is obvious that the magnetic field produced due to the rings is in same direction on the common axis. Moreover the lines close to the axis are almost parallel to the axis and in the same direction.

Thus if a number of such rings are kept very closed to each other and current is passed in the same direction.



The magnetic fields associated with each single turn are almost concentric circles and hence tend to cancel between the turns. At the interior midpoint, the field is strong and along the axis (i.e) the field is parallel to the axis. For a point such as P, the field due to the upper part of the solenoid turns tends to cancel the field due to the lower part of the turns, acting in opposite directions. Hence the field outside the circular coil is very less Solenoid is a device in which this situation is realized.
A helical coil consisting of closely wound turns of insulated conducting wire is called solenoid.
When length of a solenoid is very large compared to its radius, the solenoid is called long solenoid. For long solenoid magnetic field outside is practically zero.

Magnetic field at a point on the axis of the a SHORT solenoid



Consider a solenoid of length L and radius R containing N closely spaced turns and carrying steady current I. let number of turns per unit length be n
The field at point P on the axis of a solenoid can be obtained by superposition of fields due to large number of turns all having their centre on the axis of the solenoid as shown in figure
Consider a coil of width dx at a distance x from the point P on the axis as shown in figure
The field at P due to ndx turns is

$$dB = \frac{\mu_0}{2}\frac{(ndx)IR^2}{\left(R^2+x^2\right)^{3/2}}$$

From figure x = R tanϕ
dx = $R\sec^2 \phi \, d\phi$
On substituting values in above equation we get

$$dB = \frac{\mu_0}{2}\frac{\left(nR\sec^2\varphi d\varphi\right)IR^2}{\left(R^2 + R^2\tan^2\varphi\right)^{3/2}}$$

$$dB = \frac{\mu_0}{2}\frac{nI\sec^2\varphi}{\sec^3\varphi}d\varphi$$

$$dB = \frac{\mu_0}{2}nI\cos\varphi d\varphi$$

$$B = \int_{-\alpha}^{\beta}\frac{\mu_0}{2}nI\cos\varphi d\varphi$$

$$B = \frac{\mu_0}{2}nI\int_{-\alpha}^{\beta}\cos\varphi d\varphi$$

$$B = \frac{\mu_0}{2}nI\left[\sin\alpha + \sin\beta\right]$$

Case (I) If the solenoid is of infinite length and the point is well inside the solenoid

In this case α = β = π/2 then B is

$$B = \frac{\mu_0}{2}nI\left[\sin\frac{\pi}{2} + \sin\frac{\pi}{2}\right]$$

$$B = \mu_0 nI$$

Case(II) If the solenoid is of INFINITE length and the point is near one end

In this case α = 0 and β = π/2

$$B = \frac{\mu_0}{2}nI\left[\sin 0 + \sin\frac{\pi}{2}\right]$$

$$B = \frac{\mu_0}{2}nI$$

Case (III) If the solenoid is of FINITE length and the point is on the perpendicular bisector of its axis

In this case α = β

$$B = \frac{\mu_0}{2}nI\left[\sin\alpha + \sin\alpha\right]$$

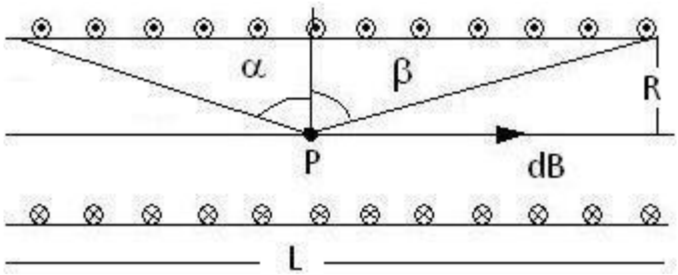
$$B = \mu_0 nI\sin\alpha$$

$$\sin\alpha = \frac{L}{\sqrt{L^2 + 4R^2}}$$

Solved problem

Q) A solenoid of length 0.4m and diameter 0.6m consists of a single layer of 1000turns of fine wire carrying a current of 5 ×10^{-3}ampere. Calculate the magnetic field on the axis of the middle and at the end of the solenoid

<u>Solution:</u>



In case of a finite solenoid, the field at the point on the axis is given by

$$B = \frac{\mu_0}{2} nI\left[\sin\alpha + \sin\beta\right]$$

$$n = \frac{N}{L} = \frac{1000}{0.4} = 2.5\times10^3$$

$$B = 2.5\pi\times10^{-6}\left[\sin\alpha + \sin\beta\right]$$

a) Middle point α = β thus
   B = 2.5π×10^{-6}(2sinα) and

$$\sin\alpha = \frac{L}{\sqrt{L^2 + 4R^2}}$$

$$\sin\alpha = \frac{L}{\sqrt{L^2 + d^2}}$$

$$\sin\alpha = \frac{0.4}{\sqrt{(0.4)^2 + (0.6)^2}}$$

$$\sin\alpha = \frac{4}{7.2}$$

   B = 2.5π×10^{-6}×2×(4/7.2) = 8.72×10^{-6} T

b) When the points is at the end on axis

$$\sin\beta = \frac{L}{\sqrt{L^2 + R^2}}$$

$$\sin\beta = \frac{0.4}{\sqrt{(0.4)^2 + (0.3)^2}}$$

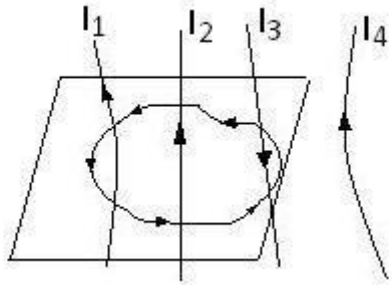
$$\sin\beta = \frac{4}{5}$$

   B = 2.5π×10^{-6}×2×(4/5) = 6.28×10^{-6} T

Ampere's Law

Consider electric currents I_1,I_2, I_3, I_4 as shown in figure. All these current produce magnetic field in the region around electric current.

A plane which is not necessarily horizontal is shown in figure. An arbitrary closed curve is also shown in figure



Sign convention for current:

Arrange right hand screw perpendicular to plane containing closed loop and rotate in the direction of vector element taken for line integration. Electric current in the direction of advancement of screw is considered positive and current in opposite direction are considered negative.

Now from sign convention I_1 and I_2 are positive while I_3 is negative.

Hence algebraic sum $\sum I = I_1 + I_2 - I_3$

Here we don't worry about current not enclosed by the loop

The statement of the Ampere's Law is as under:

The line integral of magnetic induction over a closed loop in a magnetic field is equal to the product of algebraic sum of electric current enclosed by loop and the magnetic permeability"

The law can be represented mathematically as

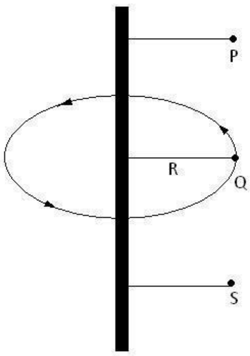
$$\oint \vec{B} \cdot \vec{dl} = \mu_0 \sum I$$

The magnetic induction in the above equation is due to all current. Whereas algebraic sum of current on right hand side is only of those currents which are enclosed by the loop

This law is true for steady current.

Application of Ampere's Law

(A) To find Magnetic field Due to a very long straight conductor carrying electric current



Let I be the current flowing through a very long conductor. Now consider points like P, Q, S located at same perpendicular distance R from wire. Since the two ends of wire at infinity and due to symmetry of wire magnetic field at pints P, Q and S is same.

Thus magnetic field at all point on the circumference of circle of radius R must be same. Or B is constant

Consider a small segment of length dl along the circumference.

Now by applying Ampere's Law we get

$$\oint \vec{B}\cdot\vec{dl} = \mu_0 \sum I$$

$$\oint B\, dl \cos\theta = \mu_0 I$$

$\vec{B}$ *and dl are in same direction*

$$\oint B\, dl = \mu_0 I$$

$$B\oint dl = \mu_0 I$$

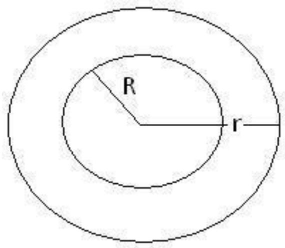
$$B(2\pi R) = \mu_0 I$$

$$B = \frac{\mu_0}{2\pi}\frac{I}{R}$$

Thus Outside the conductor B ∝ (1/R)

Magnetic field inside the conductor

Consider a top view of conductor of radius r . We want to find magnetic field at a distance R < r. Consider a loop of radius R as shown in figure



Let conductor carries a current I thus current through conductor of radius R is

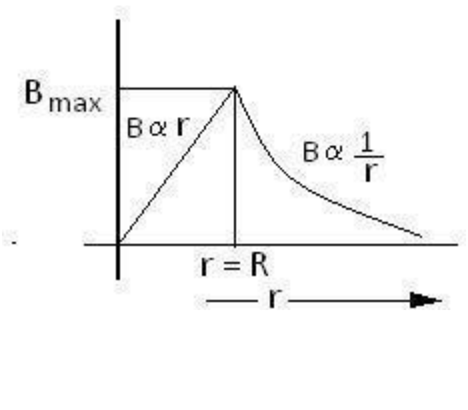
$$i = \left(\frac{I}{\pi r^2}\right)\pi R^2 = I\frac{R^2}{r^2}$$

From Ampere's Law

$$B(2\pi R) = \mu_0 i$$

$$B(2\pi R) = \mu_0 I\frac{R^2}{r^2}$$

$$B = \left(\frac{\mu_0 I}{2\pi r^2}\right)R$$

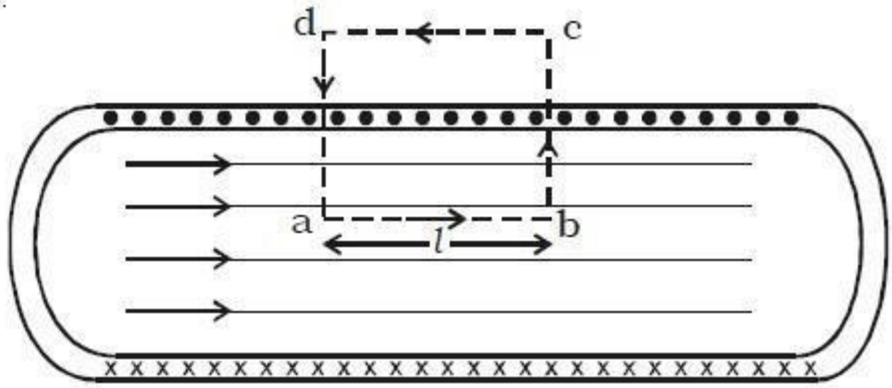


For inside the conductor B∝ R

Hence for magnetic field

(i) Outside the conductor B ∝ (1/R)
(ii) Inside the conductor B ∝ R
(iii) On the conductor Maximum
(iv) At end points outside conductor = 0

(B) Magnetic field inside a LONG solenoid using Ampere's Circuital Law



A solenoid is a wire wound closely in the form of a helix, such that adjacent turns are electrically insulated

The magnetic field inside a very tightly wound long solenoid is uniform everywhere along the axis of the solenoid and is zero outside it.

To calculate the magnetic field at point 'a', let us draw rectangle abcd as shown in figure.

The line ab is parallel to the solenoid axis and hence parallel to magnetic field **B** inside the solenoid thus **B.dl** = B(dl)

Line da and bc are perpendicular thus **B.dl** = 0

Line cd is outside the solenoid here B = 0 thus **B.dl** = 0

If i is the current and n is the number of turns per unit length then current enclosed by the loop = ni l

From Ampere's Law

$$\int_a^b B\,dl = \mu_0 nil$$

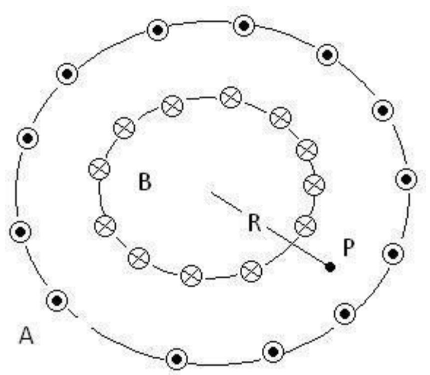
$$Bl = \mu_0 nil$$

$$B = \mu_0 ni$$

Toroid:

If a solenoid is bent in the form of a circle and its two ends are joined with each otherthe device is called a toroid.

The toroid is a hollow circular ring on which a large number of turns of a wire are closely wound. It can be viewed as a solenoid which has been bent into a circular shape to close itself. It is shown in figure carrying a current i



Magnetic field at point A and B is zero as points are outside the toroid
Magnetic field at point P inside the toroid which is at distance R from its centre as shown in figure. Clearly magnetic field at point on the circle of radius R is constant. And directing towards the tangent to the circle. Therefore

$$\oint \vec{B}\cdot\vec{dl} = B(2\pi R)$$

If total number of turns is N and current passing is I, the total current passing through said loop must be NI
From Ampere's Law

$$\oint \vec{B}\cdot\vec{dl} = \mu_0 NI$$

$$B(2\pi R) = \mu_0 NI$$

$$B = \frac{\mu_0 NI}{2\pi R}$$

$$B = \mu_0 nI$$

Here n = N/2πR the number of turns per unit length of toroid
This magnetic field is uniform at each point inside toroid
In an ideal toroid, the turns are completely circular. In such toroid magnetic field inside the toroid is uniform and outside is zero.
But toroid used in practice turns are helical and hence small magnetic field is produced outside the toroid
Toroid is used for nuclear fusion devise for confinement of plasma.

SECTION II

Force on a charged particle in a magnetic field

When a charged 'q' moving in a magnetic field **B** with velocity **v** then force experienced by the charge is given by

F= q(**v** × **B**)

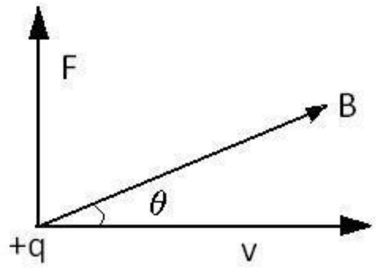
The magnitude is given by

F = qvBsinθ Here θ is angle between direction of **B** and direction of velocity **v**

Direction of force is perpendicular to both **B** and **v**

The right hand thumb rule:

For determining the direction of cross product **v** × **B**, you point the four fingers of your right hand along the direction of v, and palm in the direction of magnetic field B the curl the fingers. The thumb is then points in the direction of **v** × **B**



If q is negative then direction of F will be opposite to direction of **v** × **B**

Important points:

(1) The magnetic force will be maximum when sinθ = 1 ⟹ θ = 90°
Change is moving perpendicular to magnetic field F_{max} = qvB
In this situation F, v, B are mutually perpendicular to each other.

(2) The magnetic force will be minimum when sinθ = 0 ⟹ θ = 0 or 180°
It means charge is moving parallel to magnetic field F_{min} = 0

(3) Magnetic force is zero when charge is stationary

(4) In case of motion of charged particle in a magnetic field, as the force is always perpendicular to direction of charge work done is zero. Or magnetic force cannot change kinetic energy of charge and speed remains constant

Difference between Electric and Magnetic field

(1) Magnetic force is always perpendicular to the field while electric force is collinear with the field

(2) Magnetic force is velocity dependent i.e. acts only when charged particle is in motion while electric force is independent of the state of rest or motion of the charge.

(3) Magnetic force does not work when the charged particle is displaced while electric force does work in displacing the charged particle.

(4) Magnetic force is always non-central while the electric force may or may not be.

Non-central force: A force between two particles that is not directed along the line connecting them.

Motion of a charged particle in a uniform magnetic field

(A) When the charged particle is given velocity perpendicular to the field:

Let a particle of charge q and mass m is moving with a velocity 'v' and enters at right angles to uniform magnetic field N as shown in figure

The force on the particle is qvB and this force will always act in a direction perpendicular to v. Hence, the particle will always act in a direction perpendicular to v. Hence the particle will move on a circular path. If the radius of the path is r then

$$\frac{mv^2}{r} = Bqv$$

$$r = \frac{mv}{qB}$$

Thus radius of the path is proportional to the momentum mv of the particle and inversely proportional to the magnitude of magnetic field

Time period:

The time period is the time taken by the charged particle to complete on rotation of the circular path which is given by

$$T = \frac{2\pi r}{v} = \frac{2\pi m}{qB}$$

The time period is independent of the speed

Frequency:

The frequency is the number of revolution of charged particle in one second which is given by

$$f = \frac{1}{T} = \frac{qB}{2\pi m}$$

SOLVED NUMERICAL

Q) Two particles of mass M and m and having equal electric charge are accelerated through equal potential difference and then move inside a uniform magnetic field, normal to it. If the radii of the circular paths are R and r respectively find the ratio of their masses

Solution:

Since charge is same on both the particles and are accelerated through equal potential both particles will have same kinetic energy. Let p_1 be the momentum of particle of mass M and p_2 be the momentum of particle of mass 'm' thus

$$\frac{p_1^2}{2M} = \frac{p_2^2}{2m}$$

$$\frac{p_1^2}{p_2^2} = \frac{M}{m}$$

From the equation for radius r ∝ momentum

Thus

$$\frac{M}{m} = \left(\frac{R}{r}\right)^2$$

(B) When a charged particle is moving at an angle to the field

In this case the charged particle having charge q and mass m is moving with velocity v and it enters the magnetic field B at angle θ as shown in figure. Velocity can be resolved in two component one along magnetic field and the other perpendicular to it. Let these components are $V_{||}$ and $V_{\perp}$

$V_{||} = V\cos\theta$ and $V_{\perp} = V\sin\theta$

The parallel component $V_{||}$ of velocity remains unchanged as it is parallel to B.

Due to perpendicular component $V_{\perp}$ the particle will move on a circular path.

So resultant path will be combination of straight line motion and circular motion, which will be helical path

The radius of path:

$$r = \frac{mv\sin\theta}{qB}$$

Time period:

$$T = \frac{2\pi r}{v_{\perp}}$$

$$T = \frac{2\pi mv\sin\theta}{v\sin\theta qB}$$

$$T = \frac{2\pi m}{qB}$$

Frequency (f)

$$f = \frac{qB}{2\pi m}$$

Pitch :

Pitch of helix described by charged particle is defined as the distance moved by the centre of circular path in the time in which particle completes one revolution

Pitch = $V_{||}$ (time period)

$$pitch = v\cos\theta\frac{2\pi m}{Bq}$$

$$pitch = \frac{2\pi mv\cos\theta}{Bq}$$

Motion of charged particle in combined electric and magnetic field

When the moving charged particle is subjected simultaneously to both electric field and magnetic field B, the moving charged particle will experience electric force $\mathbf{F_e} = q\mathbf{E}$ and magnetic force $\mathbf{F_m} = q(\mathbf{v} \times \mathbf{B})$ **,** so the net force on it will be

$\mathbf{F} = q[E + (\mathbf{v} \times \mathbf{B})]$ which is called Lorentz force equation**:**

Case (I) when v, B, and E all three are collinear

In this situation as the particle is moving parallel or antiparallel to the field, the magnetic force on it will be zero and only electric force will act so

$\vec{a} = \frac{q\vec{E}}{m}$

Hence particle will flow straight path with changing speed and hence kinetic energy, momentum will also change

Case(II) **v, E** and **B** are mutually perpendicular

v, E and **B** are mutually perpendicular.

In case situation of E and B are such that

F = $\mathbf{F_e} + \mathbf{F_m} = 0$

Then a = 0, particle will move in its original without change in velocity in this situation

qE = qvB

or v = E/B

This principle is used in velocity selector to get a charged beam having a specific velocity

SOLVED PROBLEM

Q) A particle of mass 1×10^{-26} kg and charge $+1.6\times10^{-19}$ C travelling with velocity 1.28×10^{6} m/s in +x direction enters a region in which a uniform electric field E and a uniform magnetic field b are present such that $E_x = E_y = 0$; $E_z = 102.4$ kV/m and $B_x = B_z = 0$, $B_z = 8\times10^{-2}$ wb/m^2. The particle enters in a region at the origin at time t = 0. Find the location (x, y, z) of the particle at t = 5×10^{-6}s

Solution

From Lorentz equation

$\mathbf{F} = q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})]$

$F = q[102.4 \times10^{3}\ \mathbf{i} + (1.28\times10^{6}\ \mathbf{i} \times 8\times10^{-2}\ \mathbf{k})]$

$F = q[102.4 \times10^{3}\ \mathbf{i} + (-102.4 \times10^{3}\ \mathbf{i})]$

$\mathbf{F} = 0$

Hence, the particle will move along + x axis with constant velocity 1.28×10^{6} m/s

X=vt = 6.40 m

Location is (6.4, 0, 0)

Cyclotron

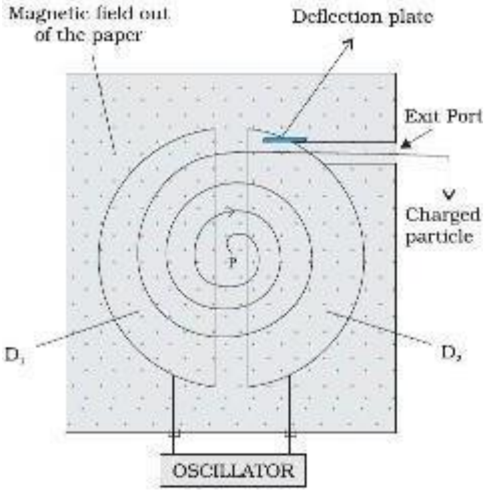
Cyclotron is a device used to accelerate charged particles to high energies. It was devised by Lawrence.

Principle

Cyclotron works on the principle that a charged particle moving normal to a magnetic field experiences magnetic Lorentz force due to which the particle moves in a circular path.

Construction

It consists of a hollow metal cylinder divided into two sections D_1 and D_2 called Dees, enclosed in an evacuated chamber. The Dees are kept separated and a source of ions is placed at the centre in the gap between the Dees. They are placed between the pole pieces of a strong electromagnet. The magnetic field acts perpendicular to the plane of the Dees. The Dees are connected to a high frequency oscillator. The whole assembly is evacuated to minimize collisions between the ions and the air molecules. A high frequency alternating voltage is applied to the Dees. In the sketch shown in Fig. positive ions or positively charged particles (e.g., protons) are released at the centre P.



Working

When a positive ion of charge *q* and mass *m* is emitted from the source, it is accelerated towards the Dee having a negative potential at that instant of time. Due to the normal magnetic field, the ion experiences magnetic Lorentz force and moves in a circular path. By the time the ion arrives at the gap between the Dees, the polarity of the Dees gets reversed. Hence the particle is once again accelerated and moves into the other Dee with a greater velocity along a circle of greater radius. Thus the particle moves in a spiral path of increasing radius and when it comes near the edge, it is taken out with the help of a deflector plate (D.P). The particle with high energy is now allowed to hit the target. When the particle moves along a circle of radius *r* with a velocity *v*, the magnetic Lorentz force provides the necessary centripetal force. They move in a semi-circular path in one of the dees and arrive in the gap between the dees in a time interval *T*/2; where *T,* the period of revolution

$$Bqv = \frac{mv^2}{r}$$

$$r = \frac{mv}{Bq}$$

The time taken to describe a circle

$$r = \frac{mv}{Bq}$$

$$r = \frac{m\omega r}{Bq}$$

$$\omega = \frac{Bq}{m}$$

$$But\ \omega = \frac{2\pi}{T}$$

$$\frac{2\pi}{T} = \frac{Bq}{m}$$

$$T = \frac{2\pi m}{Bq}$$

$$f = \frac{Bq}{2\pi m}$$

It is clear from equation that the time taken by the ion to describe a circle is independent of (i) the radius (r) of the path and (ii) the velocity (*v*) of the particle
So, in a uniform magnetic field, the ion traverses all the circles in exactly the same time. If the high frequency oscillator is adjusted to produce oscillations of frequency as given in equation resonance occurs. Cyclotron is used to accelerate protons, deutrons and α - particles.
Limitations
(i) Maintaining a uniform magnetic field over a large area of the Dees is difficult.
(ii) At high velocities, relativistic variation of mass of the particle upsets the resonance condition.
(iii) At high frequencies, relativistic variation of mass of the electron is appreciable and hence electrons cannot be accelerated by cyclotron.

Solved numerical

Q) A particle having 2C charge passes through magnetic field of 4 k T and some uniform electric field with velocity 25j. IF Lorentz force acting on it is 400 I N. find the electricfield in this region
Solution
Lorentz force
F =q[**E** +(**v** ×**B**)]
400 **i** = 2 [E + 25(4) (**j** ×**k**)]
400 **i** = 2E +200**i**
E = 100 **i** V/m

Force on current carrying conductor in magnetic field

Let L be the length of the straight conductor carrying current I and placed perpendicular to a uniform magnetic induction B

A current in a conductor is due to flow of charge

If v_d is drift velocity of charge A is cross section of conductor n is density of charge per unit volume then from equation

$I = nqv_dA$

Now number of charges in conductor of length L is N = nAL

The force on N charges $F = BNqv_d$

Total force $F = NnALqv_d$

From equation for current

F = BIL

In vector form **F** = I (**L** × **B**)

Where L is a vector in the direction of the current, magnitude of **L** is L for the segment of wire in a uniform magnetic field

F = ILBsinθ here θ is the angle between vector IL and **B**

Magnitude of the force

The magnitude of the force is F = BIL sin θ

(i) If the conductor is placed along the direction of the magnetic field, θ = 0^o, Therefore force F = 0.

(ii) If the conductor is placed perpendicular to the magnetic field,
θ = 90^o, F = BI*l.* Therefore the conductor experiences maximum force.

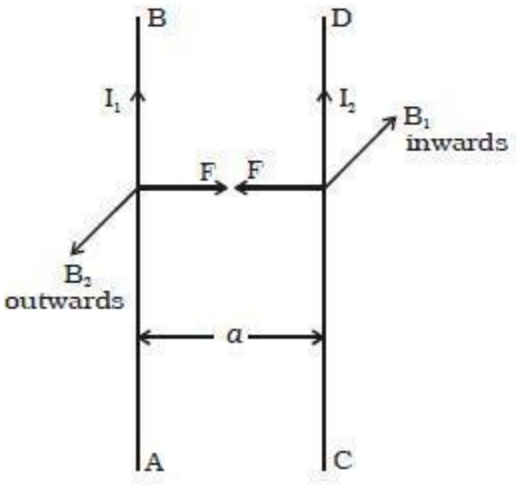
(iii) Force on a closed loop of an arbitrarily shaped conductor is zero

The direction of the force on a current carrying conductor placed in a magnetic field is given by Fleming's Left Hand Rule.

The forefinger, the middle finger and the thumb of the left hand are stretched in mutually perpendicular directions. If the forefinger points in the direction of the magnetic field, the middle finger points in the direction of the current, then the thumb points in the direction of the force on the conductor.

Force between two long straight parallel current carrying conductors

AB and CD are two straight very long parallel conductors placed in air at a distance *a*. They carry currents I_1 and I_2 respectively.



The magnetic induction due to current I_1 in AB at a distance *a* is

$$B_1 = \frac{\mu_0 I_1}{2\pi a} \quad \text{...eq(1)}$$

This magnetic field acts perpendicular to the plane of the paper and inwards. The conductor CD with current I_2 is situated in this magnetic field. Hence, force on a segment of length L of CD due to magnetic field B_1 is

$F = B_1 I_2 L$

substituting equation (1)

$$F = \frac{\mu_0 I_1 I_2 L}{2\pi a}$$

By Fleming's Left Hand Rule, F acts towards left. Similarly, the magnetic induction due to current I_2 flowing in CD at a distance *a* is

$$B_2 = \frac{\mu_0 I_2}{2\pi a} \quad \text{.....eq(3)}$$

This magnetic field acts perpendicular to the plane of the paper and outwards. The conductor AB with current I1, is situated in this field. Hence force on a segment of length *l* of AB due to magnetic field B2 is

$$F = B_2 I_1 L$$

substituting equation (3)

$$F = \frac{\mu_0 I_1 I_2 L}{2\pi a}$$

By Fleming's left hand rule, this force acts towards right. These two forces given in equations (2) and (4) attract each other. Hence, two parallel wires carrying currents in the same direction attract each other and if they carry currents in the opposite direction, repel each other.

Definition of ampere

The force between two parallel wires carrying currents on a segment of length L is

$$F = \frac{\mu_0 I_1 I_2 L}{2\pi a}$$

Force per unit length of the conductor is

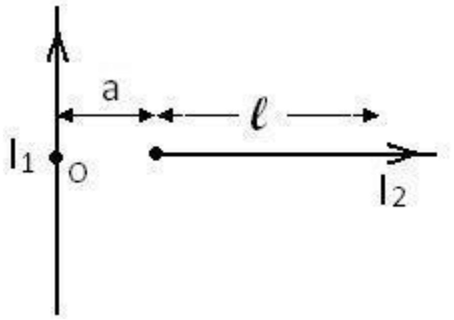
$$\frac{F}{L} = \frac{\mu_0 I_1 I_2}{2\pi a}$$

If $I_1 = I_2 = 1$ Amp , a = 1 m Then $F/L = 2 \times 10^{-7}$

The above conditions lead the following definition of ampere. Ampere is defined as that constant current which when flowing through two parallel infinitely long straight conductors of negligible cross section and placed in air or vacuum at a distance of one metre apart, experience a force of 2×10^{-7} newton per unit length of the conductor.

Solved Numerical

Q) As shown in figure very long conductor wire carrying current I_1 is arranged in y direction another conducting wire of length l carrying current I_2 is placed along X-axis at a distance a from this wire. Find the torque acting on this wire with respect to point O



Solution:

We can consider wire of current I_2 is in the magnetic field produced current I_1

The force acting on a current element I_2dx located at a distance x from O is,

$dF = I_2dx\, \mathbf{l} \times \mathbf{B}$

Here B is

$$B = \frac{\mu_0 I_1}{2\pi x}\left(-\hat{k}\right)$$

Thus

$$d\vec{F} = I_2 dx\hat{i} \times \frac{\mu_0 I_1}{2\pi x}\left(-\hat{k}\right)$$

$$d\vec{F} = \frac{\mu_0 I_1 I_2 dx}{2\pi x}\hat{j}$$

The torque acting on this element with respect to O is

$d\tau = x\mathbf{i} \times d\mathbf{F}$

$$d\vec{\tau} = x\hat{i} \times \frac{\mu_0 I_1 I_2 dx}{2\pi x}\hat{j}$$

$$d\vec{\tau} = \frac{\mu_0 I_1 I_2}{2\pi} dx\hat{k}$$

Total torque acting on this coil can be obtained by taking integration of this equation between x = 0 to x = a + l

$$\vec{\tau} = \frac{\mu_0 I_1 I_2}{2\pi} \int_a^{a+l} dx\hat{k}$$

$$\vec{\tau} = \frac{\mu_0 I_1 I_2}{2\pi} [x]_a^{a+l}$$

$$\vec{\tau} = \frac{\mu_0 I_1 I_2}{2\pi} [a + l - a]\hat{k}$$

$$\vec{\tau} = \frac{\mu_0 I_1 I_2 l}{2\pi} \hat{k}$$

Q) A straight wire of length 30cm and mass 60mg lies in a direction 30^o east of north. The earth's magnetic field at this is in horizontal and has a magnitude of 0.8 ×10^{-4}T. What current must be passes through the wire so that it may float in air? [g = 10 m/s^2]

Solution:

$$F = BIL\sin\theta$$

This force will act upward should be equal to downward force of gravitation = mg thus

$$mg = BIL\sin\theta$$

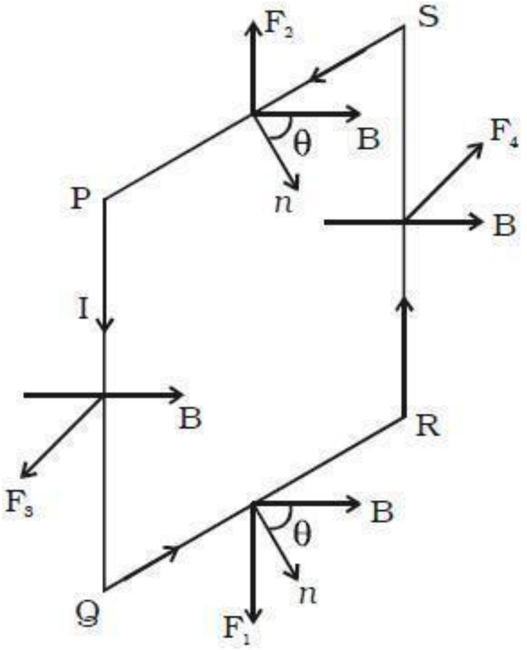
m = 60×10^{-6}Kg , B= 0.8 ×10^{-4}T, L = 30×10^{-2}, θ = 30^o, g = 10 m/s^2

60×10^{-6}×10 = 0.8 ×10^{-4} × (I) × 30×10^{-2}× (1/2)

I = 50 A

Current Loop in uniform Magnetic field

Let us consider a rectangular loop PQRS of length *l* and breadth *b* (Fig 3.24). It carries a current of I along PQRS. The loop is placed in a uniform magnetic field of induction B. Let θ be the angle between the normal to the plane of the loop and the direction of the magnetic field. Force on the arm QR,



Force on the arm QR,

$$\vec{F_1} = \overline{I(QR)} \times \vec{B}$$

Since the angle between **I(QR)** and **B** is (90^o – θ),
Magnitude of the force F$_1$ = BIb sin (90^o – θ)

ie. F$_1$ = BIb cos θ

Force on the arm SP,

$$\vec{F_2} = \overline{I(SP)} \times \vec{B}$$

Since the angle between **I(SP)** and **B** is (90^o + θ),
Magnitude of the force F$_2$ = BIb cos θ
The forces F$_1$ and F$_2$ are equal in magnitude, opposite in direction and have the same line of action. Hence their resultant effect on the loop is zero.
Force on the arm PQ,

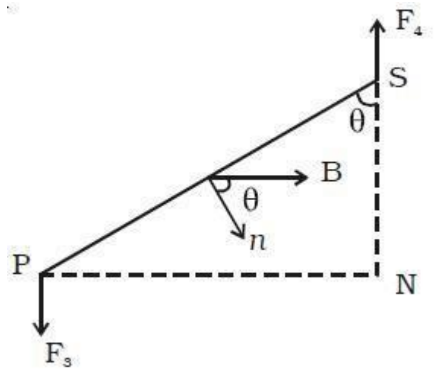
$$\vec{F} = \overline{I(PQ)} \times \vec{B}$$

Since the angle between I(PQ) and B is 90°,
Magnitude of the force F_3 = BIL sin 90° = BIL
F_3 acts perpendicular to the plane of the paper and outwards.

Force on the arm RS,

$$\overrightarrow{F_4} = \overrightarrow{I(RS)} \times \overline{B}$$

Since the angle between I(RS) and B is 90°,
Magnitude of the force F_4 = BIL sin 90o = BIL



F_4 acts perpendicular to the plane of the paper and inwards.

The forces F_3 and F_4 are equal in magnitude, opposite in direction and have different lines of action. So, they constitute a couple. Hence,
Torque = BIL × PN = BIL × PS × sin θ = BIL × b sin θ = BIA sin θ
If the coil contains N turns, τ = NBIA sin θ
So, the torque is maximum when the coil is parallel to the magnetic field and zero when the coil is perpendicular to the magnetic field.
The torques can be expressed as vector product of the magnetic moment of the coil and the magnetic field. We define the magnetic moment of the current loop as, **m** =N I **A** where the direction of the area vector **A** is given by the right-hand thumb rule and is directed into the plane of the paper in Figure. Then as the angle between **m** and **B** is θ
τ = **m** × **B**
The dimensions of the magnetic moment are [A] [L^2] and its unit is Am^2.
From equation we see

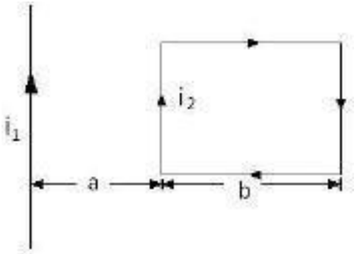
(i) the torque τ vanishes when **m** is either parallel or antiparallel to the magnetic field **B**.
(ii) This indicates a state of equilibrium as there is no torque on the coil (this also applies to any object with a magnetic moment **m**).
(iii) When **m** and **B** are parallel the equilibrium is a stable one. Any small rotation of the coil produces a torque which brings it back to its original position.
(iv) When they are antiparallel, the equilibrium is unstable as any rotation produces a torque which increases with the amount of rotation.
(v) The presence of this torque is also the reason why a small magnet or any magnetic dipole aligns itself with the external magnetic field.

Solved Numerical

Q) The arrangement is as shown below

(a) Find the potential energy of the loop

(b) Find the work done to increase the spacing between the wire and the loop a to 2a

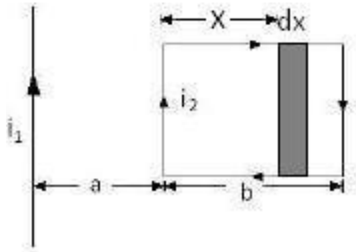


Solution:

(a) Magnetic field produced due to wire carrying current I_1 is inversely proportional to distance thus magnetic field associated with loop is not uniform. Consider small area of width dx and height L

Magnetic moment of a small element of the loop dM = i_2Ldx

The direction of the magnetic moment is perpendicular to the plane of the paper



Potential energy dU = -**dM** . **B**

Where B is the magnetic field at the position of this element

$$B = \frac{\mu_0}{4\pi}\frac{2I_1}{a+x}$$

$$dU = -\frac{\mu_0}{4\pi}2I_1I_2L\left(\frac{dx}{a+x}\right)$$

$$U = -\frac{\mu_0}{4\pi}2I_1I_2L\int_0^L\left(\frac{dx}{a+x}\right)$$

$$U = \frac{\mu_0}{4\pi}2I_1I_2L\log\left(\frac{a+b}{a}\right)$$

(b) Work done to increase the spacing between the wire and the loop from a to 2a

W = ΔU

$$U_i = -\frac{\mu_0}{4\pi} 2I_1I_2L\log\left(\frac{a+b}{a}\right)$$

$$U_f = \frac{\mu_0}{4\pi} 2I_1I_2L\log\left(\frac{2a+b}{2a}\right)$$

$$\Delta U = U_f - U_i = \frac{\mu_0}{4\pi} 2I_1I_2L\log\left(\frac{2a+2b}{2a+b}\right)$$

Q) A rectangular coil of area 20 cm × 10 cm with 100 turns of wire is suspended in a radial magnetic field of induction 5 × 10-3 T. If the galvanometer shows an angular deflection of 150 for a current of 1mA, find the torsional constant of the suspension wire.

Solution:

n = 100, A = 20 cm × 10 cm = 2 × 10^{-1} × 10^{-1} m^2 , B = 5 × 10^{-3} T, , I = 1mA = 10^{-3}A,

θ = 15^0 = 15 (π/180) = π/12 , C = ?

nBIA = Cθ

$$C = \frac{nBIA}{\theta}$$

$$C = \frac{10\times5\times10^{-3}\times10^{-3}\times2\times10^{-1}\times10^{-1}}{\left(\frac{\pi}{12}\right)}$$

$$C = 3.82\times10^{-5}\, Nm\, rad^{-1}$$

Moving coil galvanometer

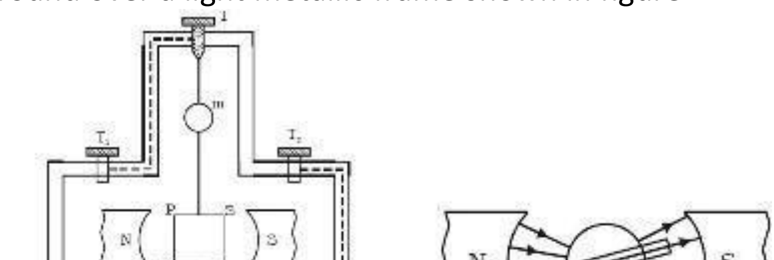
Moving coil galvanometer is a device used for measuring the current in a circuit.

Principle

Moving coil galvanometer works on the principle that a current carrying coil placed in a magnetic field experiences a torque.

Construction

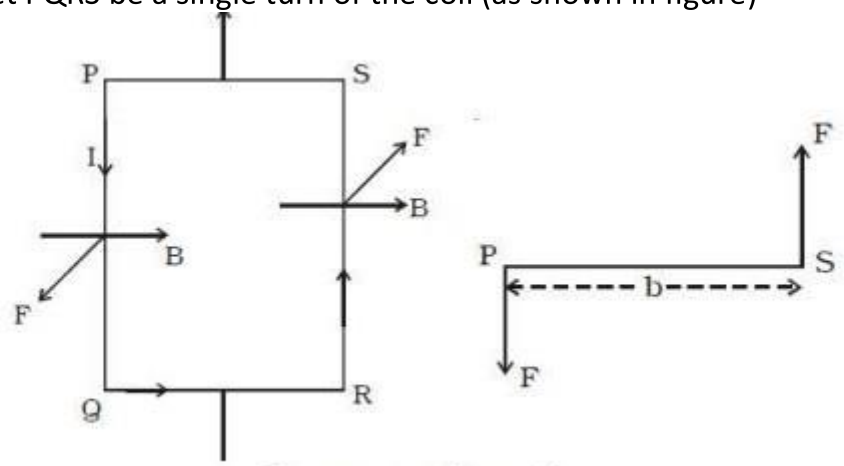
It consists of a rectangular coil of a large number of turns of thin insulated copper wire wound over a light metallic frame shown in figure



The coil is suspended between the pole pieces of a horse-shoe magnet by a fine phosphor – bronze strip from a movable torsion head. The lower end of the coil is connected to a hair spring (HS) of phosphor bronze having only a few turns. The other end of the spring is connected to a binding screw. A soft iron cylinder is placed symmetrically inside the coil. The hemispherical magnetic poles produce a radial magnetic field in which the plane of the coil is parallel to the magnetic field in all its positions as shown in figure. A small plane mirror (m) attached to the suspension wire is used along with a lamp and scale arrangement to measure the deflection of the coil.

Theory

Let PQRS be a single turn of the coil (as shown in figure)



Torque on the coil

A current I flows through the coil. In a radial magnetic field, the plane of the coil is always parallel to the magnetic field. Hence the sides QR and SP are always parallel to the field. So, they do not experience any force. The sides PQ and RS are always perpendicular to the field. PQ = RS = L, length of the coil and PS = QR = b, breadth of the coil

Force on PQ, F = BI (PQ) = BIL. According to Fleming's left hand rule, this force is normal to the plane of the coil and acts outwards.

Force on RS, F = BI (RS) = BIL.

This force is normal to the plane of the coil and acts inwards.

These two equal, oppositely directed parallel forces having different lines of action constitute a couple and deflect the coil. If there are *n* turns in the coil,

Torque of the deflecting couple = *N* BIL × b = *N*BIA

When the coil deflects, the suspension wire is twisted. On account of elasticity, a restoring couple is set up in the wire. This couple is proportional to the twist. If θ is the angular twist, then, moment of the restoring couple = Cθ

where C is the restoring couple per unit twist

At equilibrium, deflecting couple = restoring couple

$$NBIA = C\theta$$

$$I = \frac{C}{NBA}\theta$$

$$I = K\theta$$

Here K is the galvanometer constant.

$I \propto \theta$.

Since the deflection is directly proportional to the current flowing through the coil, the scale is linear and is calibrated to give directly the value of the current

We define the current sensitivity of the galvanometer as the deflection per unit current. current sensitivity is,

$$\frac{\theta}{I} = \frac{NAB}{C}$$

C is restoring couple per unit twist

Note current sensitivity is proportional to number of turns (N)

Conversion of galvanometer into an ammeter

A galvanometer is a device used to detect the flow of current in an electrical circuit. To measure current in circuit Galvanometer is connected in series. Because of following reasons Galvanometer cannot be used as Ammeter

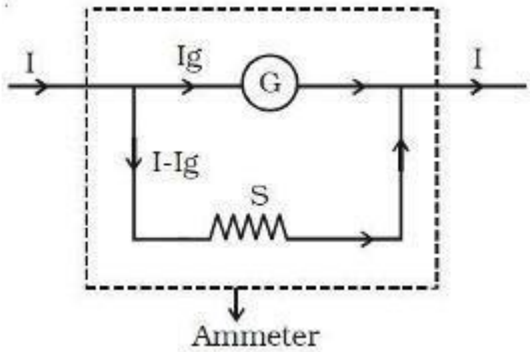
(i) Being a very sensitive instrument, a large current cannot be passed through the galvanometer, as it may damage the coil.

(ii) Galvanometer have resistance in few kilo-ohm resistance which get added to resistance of circuit as a result current in circuit changes.

However, a galvanometer is converted into an ammeter by connecting a low resistance in parallel with it. As a result, when large current flows in a circuit, only a small fraction of the current passes through the galvanometer and the remaining larger portion of the current passes through the low resistance.

The low resistance connected in parallel with the galvanometer is called shunt resistance. The scale is marked in ampere. The value of shunt resistance depends on the fraction of the total current required to be passed through the galvanometer.

Let I_g be the maximum current that can be passed through the galvanometer.



The current I_g will give full scale deflection in the galvanometer.

Galvanometer resistance = G. Shunt resistance = S

Current in the circuit = I

Current through the shunt resistance $I_s = (I - I_g)$

Since the galvanometer and shunt resistance are parallel, potential is common.

$I_g . G = (I - I_g)S$

$$S = G\frac{I_g}{I - I_g}$$

The shunt resistance is very small because Ig is only a fraction of I. The effective resistance of the ammeter R_a is (G in parallel with S)

$$\frac{1}{R_a} = \frac{1}{G} + \frac{1}{S}$$

$$R_a = \frac{GS}{G+S}$$

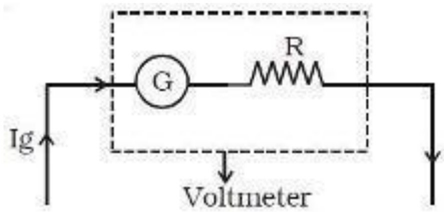
R_a is very low and this explains why an ammeter should be connected in series. When connected in series, the ammeter does not appreciably change the resistance and current in the circuit. Hence an ideal ammeter is one which has zero resistance.

Conversion of galvanometer into a voltmeter

Voltmeter is an instrument used to measure potential difference between the two ends of a current carrying conductor. A galvanometer can be converted into a voltmeter by connecting a high resistance in series with it. The scale is calibrated in volt. The value of the resistance connected in series decides the range of the voltmeter.
Galvanometer resistance = G
The current required to produce full scale deflection in the galvanometer = Ig
Range of voltmeter = V Resistance to be connected in series = R
Since R is connected in series with the galvanometer, the current through the galvanometer,

$$I_g = \frac{V}{R+G}$$

$$R = \frac{V}{I_g} - G$$



From the equation the resistance to be connected in series with the galvanometer is calculated.
The effective resistance of the voltmeter is $R_v = G + R$
R_v is very large, and hence a voltmeter is connected in parallel in a circuit as it draws the least current from the circuit.
In other words, the resistance of the voltmeter should be very large compared to the resistance across which the voltmeter is connected to measure the potential difference. Otherwise, the voltmeter will draw a large current from the circuit and hence the current through the remaining part of the circuit decreases. In such a case the potential difference measured by the voltmeter is very much less than the actual potential difference. The error is eliminated only when the voltmeter has a high resistance. An ideal voltmeter is one which has infinite resistance.

Current loop as a magnetic dipole

Ampere found that the distribution of magnetic lines of force around a finite current carrying solenoid is similar to that produced by a bar magnet. This is evident from the fact that a compass needle when similar deflections moved around these two bodies show. After noting the close resemblance between these two, Ampere demonstrated that a simple current loop behaves like a bar magnet and put forward that all the magnetic phenomena is due to circulating electric current. This is Ampere's hypothesis.

The magnetic induction at a point along the axis of a circular coil carrying current is

$$B = \frac{\mu_0 N I a^2}{2\left(a^2 + x^2\right)^{3/2}}$$

The direction of this magnetic field is along the axis and is given by right hand rule. For points which are far away from the centre of the coil, *x>>a,* a^2 is small and it is neglected. Hence for such points,

$$B = \frac{\mu_0 N I a^2}{2x^3}$$

If we consider a circular loop, n = 1, its area A = πa^2

$$B = \frac{\mu_0 I A}{2\pi x^3}$$ ....eq(1)

The magnetic induction at a point along the axial line of a short bar magnet is

$$B = \frac{\mu_0}{4\pi}\frac{2M}{x^3}$$

$$B = \frac{\mu_0}{2\pi}\frac{M}{x^3}$$ ...eq(2)

Comparing equations (1) and (2), we find that

M = IA ...(3)

Hence a current loop is equivalent to a magnetic dipole of moment M = IA

The magnetic moment of a current loop is defined as the product of the current and the loop area. Its direction is perpendicular to the plane of the loop.

The magnetic dipole moment of a revolving electron

According to Neil Bohr's atom model, the negatively charged electron is revolving around a positively charged nucleus in a circular orbit of radius *r*. The revolving electron in a closed path constitutes an electric current. The motion of the electron in anticlockwise direction produces conventional current in clockwise direction. Current, i = e/T

where T is the period of revolution of the electron. If f is the orbital velocity of the electron, then

$$T = \frac{2\pi r}{v}$$

$$i = \frac{ev}{2\pi r}$$

Due to the orbital motion of the electron, there will be orbital magnetic moment μ_l
$\mu_l = i$ A, where A is the area of the orbit.

$$\mu_l = \frac{ev}{2\pi r}\pi r^2$$

$$\mu_l = \frac{evr}{2}$$

If m is the mass of the electron

$$\mu_l = \frac{e}{2m}(mvr)$$

mvr is the angular momentum (l) of the electron about the central nucleus.

$$\mu_l = \frac{e}{2m}(l)$$

$$\frac{\mu_l}{l} = \frac{e}{2m}$$

is called gyromagnetic ratio and is a constant. Its value is 8.8×10^{10} C kg^{-1}. Bohr hypothesized that the angular momentum has only discrete set of values given by the equation.

$l = nh/2\pi$ ...(2) where n is a natural number
and h is the Planck's constant = 6.626×10^{-34} Js.
From above two equations for μ_l we get

$$\mu_l = \frac{e}{2m}\frac{nh}{2\pi} = \frac{neh}{4\pi m}$$

The minimum value of magnetic moment is

$$(\mu_l)_{min} = \frac{eh}{4\pi m}, n = 1$$

Value of (eh/4πm) is called Bohr magneton
By substituting the values of e, h and m, the value of Bohr magneton is found to be 9.27×10^{-24} Am^2
In addition to the magnetic moment due to its orbital motion, the electron possesses magnetic moment due to its spin. Hence the resultant magnetic moment of an electron is the vector sum of its orbital magnetic moment and its spin magnetic moment.

Solved Numerical

Q) A moving coil galvanometer of resistance 20 Ω produces full scale deflection for a current of 50 mA. How you will convert the galvanometer into (i) an ammeter of range 20 A and (ii) a voltmeter of range 120 V.

Solution:

G = 20 Ω ; Ig = 50 x 10^{-3} A ; I = 20 A, S = ?

V = 120 V, R = ?

(i) Ammeter

$$S = G\frac{I_g}{I - I_g}$$

$$S = 20\frac{50\times10^{-3}}{20 - 50\times10^{-3}}$$

$$S = 0.05\,\Omega$$

A shunt of 0.05 Ω should be connected in parallel

(ii) Voltmeter

$$R = \frac{V}{I_g} - G$$

$$R = \frac{120}{50\times10^{-3}} - 20$$

$$R = 2380\,\Omega$$

A resistance of 2380 Ω should be connected in series with the galvanometer

Q) The deflection in a galvanometer falls from 50 divisions to 10 divisions when 12 Ω resistance is connected across the galvanometer. Calculate the galvanometer resistance.

Solution:

θ_1 = 50 divs, θg = 10 divs, S = 12Ω G = ?

I α θ_1

Ig α θg

In a parallel circuit potential is common.

$$G\cdot I_g = S\left(I - I_g\right)$$

$$G = \frac{S\left(I - I_g\right)}{I_g}$$

$$G = \frac{12\left(50 - 10\right)}{10}$$

$$G = 48\,\Omega$$

Q) In a hydrogen atom electron moves in an orbit of radius 0.5 Å making 1016 revolutions per second. Determine the magnetic moment associated with orbital motion of the electron.

<u>Solution:</u>

$r = 0.5$ Å $= 0.5\text{x}10^{-10}$ m, $n = 1016\ s^{-1}$

Orbital magnetic moment $\mu_l = i.A$ ...(1)

$i = e/T$

$I = e.f$ ...(2)

$A = \pi r^2$ ...(3)

substituting equation (2), (3) in (1)

$\mu_l = e.n.\ \pi r^2$

$= 1.6 \times 10\text{-}19 \times 1016 \times 3.14\ (0.5 \times 10^{-10})^2$

$= 1.256 \times 10^{-23}$

$\therefore \mu_l = 1.256 \times 10^{-23}\ Am^2$

MAGNETISM AND MATTER

Bar magnet

The iron ore magnetite which attracts small pieces of iron, cobalt, nickel etc. is a natural magnet. The natural magnets have irregular shape and they are weak. A piece of iron or steel acquires magnetic properties when it is rubbed with a magnet. Such magnets made out of iron or steel are artificial magnets. Artificial magnets can have desired Shape and desired strength. If the artificial magnet is in the form of a rectangular or cylindrical bar, it is called a bar magnet.

Basic properties of magnets

(i) When the magnet is dipped in iron filings, they cling to the ends of the magnet. The attraction is maximum at the two ends of the magnet. These ends are called poles of the magnet.

(ii) When a magnet is freely suspended, it always points along north-south direction. The pole pointing towards geographic north is called North Pole *N* and the pole which points towards geographic south is called South Pole *S.*

(iii) Magnetic poles always exist in pairs. (i.e) isolated magnetic pole does not exist.

(iv) The magnetic length of a magnet is always less than its geometric length, because the poles are situated a little inwards from the free ends of the magnet. (But for the purpose of calculation the geometric length is always taken as magnetic length.)

(v) Like poles repel each other and unlike poles attract each other.

North Pole of a magnet when brought near North Pole of another magnet, We can observe repulsion, but when the north pole of one magnet is brought near South Pole of another magnet, we observe attraction.

(vi) The force of attraction or repulsion between two magnetic poles is given by Coulomb's inverse square law.

Magnetic field

Magnetic field is the space in which a magnetic pole experiences a force or it is the space around a magnet in which the influence of the magnet is felt.

Magnetic lines of force

A magnetic field is better studied by drawing as many numbers of magnetic lines of force as possible. A magnetic line of force is a line along which a free isolated north pole would travel when it is placed in the magnetic field.

Properties of magnetic lines of force

(i) Magnetic lines of forces are closed continuous curves, extending through the body of the magnet.

(ii) The direction of line of force is from North Pole to South Pole outside the magnet. While it is from South Pole to North Pole inside the magnet.

(iii) The tangent to the magnetic line of force at any point gives the direction of magnetic field at that point. (i.e) it gives the direction of magnetic induction ($\vec{B}$) at that point.

(iv) They never intersect each other.

(v) They crowd where the magnetic field is strong and thin out where the field is weak.

Magnetic moment

Since any magnet has two poles, it is also called a magnetic dipole. The magnetic moment of a magnet is defined **as the product of the pole strength and the distance between the two poles.** If m is the pole strength of each pole and 2l is the distance

between the poles, the magnetic moment Magnetic moment is a vector quantity. It is denoted by M. Its unit is A m^2. Its direction is from south pole to north pole

$$\vec{M} = m(2l)$$

Bar magnet as an equivalent solenoid

The magnetic dipole moment m associated with a current loop was defined to be m=NIA where N is the number of turns in the loop, I the current and A the area vector. The direction of magnetic moment m of a loop can be found by using right hand rule, curl fingers in the direction of current then thumb gives the direction of magnetic moment. The resemblance of magnetic field lines for a bar magnet and a solenoid suggest that a bar magnet may be thought of as a large number of circulating currents in analogy with a solenoid.
Let the solenoid consists of n turns per unit length. Let its length be 2l and radius a. We can evaluate the axial field at a point P, at a distance r from the centre O of the solenoid. To do this, consider a circular element of thickness dx of the solenoid at a distance x from its centre. It consists of n d x turns. Let I be the current in the solenoid. The magnetic field on the axis of a circular current loop at point P due to the circular element is

$$dB = \frac{\mu_0 n dx I a^2}{2[(r-x)^2 + a^2]^{3/2}}$$

The magnitude of the total field is obtained by summing over all the elements — in other words by integrating from x = – l to x = + l . Thus,

$$B = \frac{\mu_0 n I a^2}{2} \int_{-l}^{+l} \frac{dx}{2[(r-x)^2 + a^2]^{3/2}}$$

Consider the far axial field of the solenoid, i.e., r >> a and r >> l . Then the denominator is approximated by

$$2[(r-x)^2 + a^2]^{3/2} = r^3$$

Then

$$B = \frac{\mu_0 n I a^2}{2r^3} \int_{-l}^{+l} dx$$

$$B = \frac{\mu_0 n I a^2}{2r^3}(2l)$$

Note that the magnitude of the magnetic moment of the solenoid is,
m = n (2l) I (πa^2) = (total number of turns × current × cross-sectional area).
Thus,

$$B = \frac{\mu_0}{4\pi}\frac{2m}{r^3}$$

This is also the far axial magnetic field of a bar magnet which one may obtain experimentally. Thus, a bar magnet and a solenoid produce similar magnetic fields. The magnetic moment of a bar magnet is thus equal to the magnetic moment of an equivalent solenoid that produces the same magnetic field.

Magnetic induction
Magnetic induction is the fundamental character of a magnetic field at a point. Magnetic induction at a point in a magnetic field is the force experienced by unit north pole placed at that point. It is denoted by B. Its unit is N/Am.
It is a vector quantity. It is also called as magnetic flux density. If a magnetic pole of strength m placed at a point in a magnetic field experiences a force F, the magnetic induction at that point is

$$\vec{B} = \frac{F}{m}$$

Magnetic flux and magnetic flux density
The number of magnetic lines of force passing through an area A is called magnetic flux. It is denoted by φ. Its unit is Weber. It is a scalar quantity.

The number of magnetic lines of force crossing unit area kept normal to the direction of line of force is magnetic flux density. Its unit is Wb m^{-2} or Tesla.
Magnetic flux $\phi = \vec{B} \cdot A$

Uniform and non-uniform magnetic field
Magnetic field is said to be uniform if the magnetic induction has the same magnitude and the same direction at all the points in the region. It is represented by drawing parallel lines
If the magnetic induction varies in magnitude and direction at different points in a region, the magnetic field is said to be non-uniform. The magnetic field due to a bar magnet is non-uniform. It is represented by convergent or divergent lines

Force between two magnetic poles
In 1785, Coulomb made use of his torsion balance and discovered the law governing the force between the two magnetic poles.

Coulomb's inverse square law
Coulomb's inverse square law states that the force of attraction or repulsion between the two magnetic poles is directly proportional to the product of their pole strengths and inversely proportional to the square of the distance between them.
If m_1 and m_2 are the pole strengths of two magnetic poles separated by a distance of d in a medium, then
$F \propto m_1m_2$ and $F \propto \frac{1}{d^2}$

$$F \propto \frac{m_1m_2}{d^2}$$

$$F = K\frac{m_1m_2}{d^2}$$

where *K* is the constant of proportionality and

$$K = \frac{\mu}{4\pi}$$

where μ is the permeability of the medium. But $\mu = \mu_0 \times \mu_r$
μ_0 - permeability of free space or vacuum.
μ_r - relative permeability of the medium
Let $m_1 = m_2 = 1$, and d = 1 m

$$K = \frac{\mu}{4\pi}$$

In free space, $\mu_o = 4\pi \times 10^{-7}$ H m^{-1}

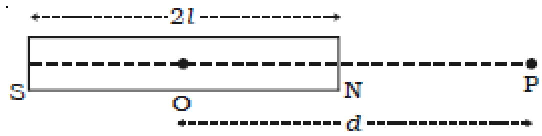
$$F = 10^{-7}\frac{m_1m_2}{d^2}$$

$$F = 10^{-7}\frac{1 \times 1}{1}$$

F = 10-7 N

Therefore, unit pole is defined as that pole which when placed at a distance of 1 metre in free space or air from an equal and similar pole, repels it with a force of 10^{-7} N.

Magnetic induction at a point along the axial line due to a magnetic dipole (Bar magnet)
NS is the bar magnet of length 2l and of pole strength m. P is a point on the axial line at a distance d from its midpoint O



According to inverse square law,

$$F = \frac{\mu_0}{4\pi}\frac{m_1 m_2}{d^2}$$

Magnetic induction (B_1) at P due to north pole of the magnet,
Along NP

$$B_1 = \frac{\mu_0}{4\pi}\frac{m}{(NP)^2}$$

$$B_1 = \frac{\mu_0}{4\pi}\frac{m}{(d-l)^2}$$

Magnetic induction (B_2) at P due to south pole of the magnet,
Along PS

$$B_2 = \frac{\mu_0}{4\pi}\frac{m}{(PS)^2}$$

$$B_2 = \frac{\mu_0}{4\pi}\frac{m}{(d+l)^2}$$

Magnetic induction at *P* due to the bar magnet,

$$B = B_1 - B_2$$

$$B = \frac{\mu_0}{4\pi}\frac{m}{(d-l)^2} - \frac{\mu_0}{4\pi}\frac{m}{(d+l)^2}$$

$$B = \frac{\mu_0 m}{4\pi}\left(\frac{4ld}{(d^2-l^2)^2}\right)$$

$$B = \frac{\mu_0 m}{4\pi}\left(\frac{2l \times 2d}{(d^2-l^2)^2}\right)$$

$$B = \frac{\mu_0}{4\pi}\left(\frac{2l \times M}{(d^2-l^2)^2}\right)$$

As $M = m \times 2l$

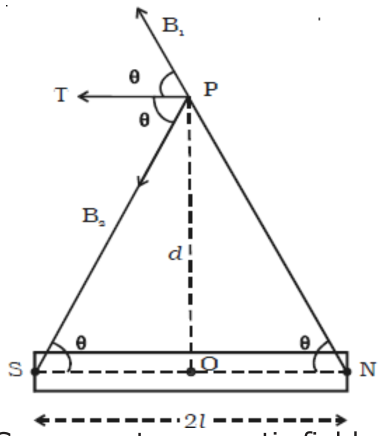
For a short bar magnet, l is very small compared to *d*, hence l^2 is neglected

$$B = \frac{\mu_0}{4\pi}\frac{2M}{d^3}$$

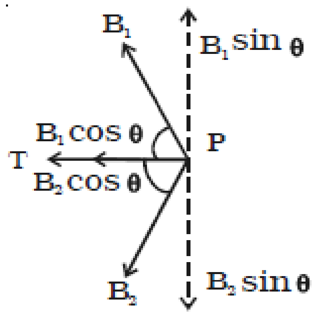
The direction of B is along the axial line away from the north pole.

Magnetic induction at a point along the equatorial line of a bar magnet

NS is the bar magnet of length 2*l* and pole strength m. P is a point on the equatorial line at a distance d from its midpoint O



Components magnetic field at point P are as follows



Magnetic induction (B_1) at P due to north pole of the magnet,
Along NP

$$B_1 = \frac{\mu_0}{4\pi}\frac{m}{(NP)^2}$$

$$B_1 = \frac{\mu_0}{4\pi}\frac{m}{(d^2+l^2)}$$

(AS $NP^2 = NO^2 + OP^2$)
Magnetic induction ($B2$) at P due to south pole of the magnet,
Along PS

$$B_2 = \frac{\mu_0}{4\pi}\frac{m}{(PS)^2}$$

$$B_2 = \frac{\mu_0}{4\pi}\frac{m}{(d^2+l^2)}$$

Resolving B_1 and B_2 into their horizontal and vertical components. Vertical components $B_1 \sin\theta$ and $B_2 \sin\theta$ are equal and opposite and therefore cancel each other
The horizontal components $B_1 \cos\theta$ and $B_2 \cos\theta$ will get added along PT.
Resultant magnetic induction at P due to the bar magnet is
$B = B_1 \cos\theta + B_2 \cos\theta$. (along PT)

$$B = \frac{\mu_0}{4\pi}\frac{m}{(d^2+l^2)}\frac{l}{\sqrt{d^2+l^2}} + \frac{\mu_0}{4\pi}\frac{m}{(d^2+l^2)}\frac{l}{\sqrt{d^2+l^2}}$$

$$cos\theta = \frac{SO}{PS} = \frac{NO}{NP}$$

As M =2lm

$$B = \frac{\mu_0}{4\pi}\frac{M}{(d^2+l^2)^{3/2}}$$

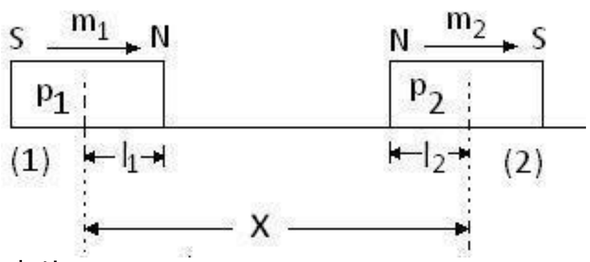
For a short bar magnet, l^2 is neglected.

$$B = \frac{\mu_0}{4\pi}\frac{M}{d^3}$$

The direction of 'B' is along PT parallel to NS.

Solved numerical

Q) Find the force between two small bar magnets of magnetic moments m_1 and m_2 lying on the axis, as shown in figure (p_1 and p_2 are the pole strength of magnet (1) and (2) respectively, X is far greater than l_1 and l_2)



Solution :

To calculate force on magnet (2) due to magnet (1)

We will calculate magnetic field due to magnet (1) at the poles of the magnet(2).

Magnet (2) is on the axis of magnet (1).

Magnetic field at North pole of magnet (2), magnetic moment of magnet(1) is m_1

$$B_N = \frac{\mu_0}{4\pi}\frac{2m_1}{(x-l_2)^3}$$

The repulsive force F_N acting on the north pole of magnet(2) having pole strength p_2

$$F_N = p_2 B_N = \frac{\mu_0}{4\pi}\frac{2m_1 p_2}{(x-l_2)^3}$$

Similarly magnetic field at South pole of magnet(2), is

$$B_S = \frac{\mu_0}{4\pi}\frac{2m_1}{(x+l_2)^3}$$

The attractive force F_S acting on the north pole of magnet(2) having pole strength p_2

$$F_S = p_2 B_S = \frac{\mu_0}{4\pi}\frac{2m_1 p_2}{(x+l_2)^3}$$

Hence resultant force on magnet(2) is

$F = F_N - F_S$

$$F = \frac{\mu_0}{4\pi} 2p_2 m_1 \left[\frac{1}{(x-l_2)^3} - \frac{1}{(x+l_2)^3}\right]$$

$$F = \frac{\mu_0}{4\pi} 2p_2 m_1 \left[\frac{6x^2 l_2}{(x^2-l_2^2)^3}\right]$$

$$F = \frac{\mu_0}{4\pi} 2p_2 m_1 \left[\frac{6x^2 l_2}{(x^2)^3}\right]$$

$$F = \frac{\mu_0 m_1}{2\pi}\left[\frac{(2p_2 l_2)(3x^2)}{x^6}\right]$$

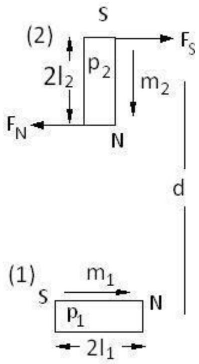
As $p_2 l_2 = m_2$

$$F = \frac{\mu_0 m_1}{2\pi}\left[\frac{(m_2)(3)}{x^4}\right]$$

$$F = \frac{3\mu_0 m_1 m_2}{2\pi x^4}$$

Resultant force is repulsive and acts on magnet (2) in a direction away from magnet(1)

Q) Find the torque on small bar magnet(2) due to small bar magnet(1), when they are placed perpendicular to each other as shown in figure l_2 and l_1 are far less than d



Solution :

From the figure Magnetic field at north pole of second magnet due to magnet(1) is

$$B_N = \frac{\mu_0}{4\pi}\frac{2m_1}{(d-l_2)^3}$$

Force on north pole is towards left is

$$F_N = p_2 B_N = \frac{\mu_0}{4\pi}\frac{2m_1 p_2}{(d-l_2)^3}$$

As X >> l_2

$$F_N = \frac{\mu_0}{4\pi}\frac{2m_1 p_2}{(d)^3}$$

Magnetic field at south pole of second magnet due to magnet(1) is

$$B_S = \frac{\mu_0}{4\pi}\frac{2m_1}{(d)^3}$$

Force on south pole is towards right is

$$F_S = p_2 B_S = \frac{\mu_0}{4\pi}\frac{2m_1 p_2}{(d+l_2)^3}$$

As X >> l_2

$$F_S = \frac{\mu_0}{4\pi}\frac{2m_1 p_2}{(d)^3}$$

Since $F_N = F_S$ are non collinear, equal and opposite in direction, they form a couple. Hence the torque is produced

$$\tau = \vec{F}_N \times 2\vec{l}_2 = \vec{F}_S \times 2\vec{l}_2$$

Since F_N and F_S are perpendicular to l_2 magnitude of torque with respect to centre of magnet (2)

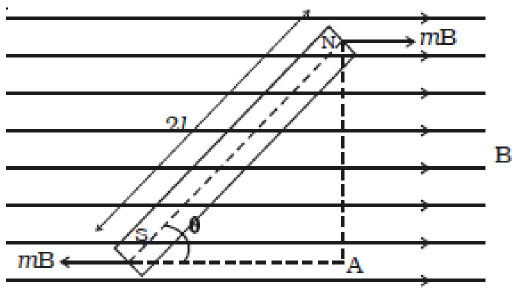
$$\tau = 2F_N l_2 = \frac{\mu_0}{4\pi}\frac{m_1 2l_2 p_2}{d^3}$$

as $2l_2p_2 = m_2$ magnetic dipole moment of magnet(2)

$$\tau = \frac{\mu_0}{4\pi}\frac{m_1 m_2}{d^3}$$

Torque on a bar magnet placed in a uniform magnetic field

Consider a bar magnet *NS* of length 2*l* and pole strength m placed in a uniform magnetic field of induction *B* at an angle θ with the direction of the field



Due to the magnetic field B, a force mB acts on the north pole along the direction of the field and a force mB acts on the south pole along the direction opposite to the magnetic field.
These two forces are equal and opposite, hence constitute a couple. The torque τ due to the couple is
τ= one of the forces × perpendicular distance between them
τ = F × NA
τ = mB × NA ...(1)
τ = mB × 2*l* sin θ
τ = MB sin θ ...(2)
Vectorially,

$$\tau = \vec{M} \times \vec{B}$$

The direction of τ is perpendicular to the plane containing $\vec{M}$ and $\vec{B}$
If B = 1 and θ = 90°
Then from equation (2), τ = M
Hence, moment of the magnet M is equal to the torque necessary to keep the agent at right angles to a magnetic field of unit magnetic induction.

Periodic time of bar magnet

Here τ is restoring torque and θ is the angle between $\vec{M}$ and $\vec{B}$.
Now Newton's second law

$$\tau = I\frac{d^2\theta}{dt^2}$$

Here I is moment of inertia of bar magnet
Therefore, in equilibrium

$$I\frac{d^2\theta}{dt^2} = -mBsin\theta$$

Negative sign with mB sinθ implies that restoring torque is in opposition to deflecting torque. For small values of θ in radians, we approximate sin θ ≈ θ and get

$$I\frac{d^2\theta}{dt^2} = -mB\theta$$

$$\frac{d^2\theta}{dt^2} = -\frac{mB}{I}\theta$$

This represents a simple harmonic motion. The square of the angular frequency is ω^2 = mB/I and the time period is

$$T = 2\pi\sqrt{\frac{I}{mB}}$$

The magnetic potential energy U_m

The magnetic potential energy U_m is given by

$$U_m = \int \tau d\theta$$

$$U_m = \int mBsin\theta d\theta$$

$$U_m = -mBcos\theta$$

$$U_m = \vec{m} \cdot \vec{B}$$

When the needle is perpendicular to the field, Equation shows that potential energy is minimum (= −mB) at θ = 0° (most stable position) and maximum (= +mB) at θ = 180° (most unstable position).

Solved numerical

Q) Work done in moving a magnet of magnetic moment m from most stable to most unstable position

<u>Solution:</u>

Most stable position is θ = 0° and most unstable position is θ = 180° hence work done
W = U_B(θ =180°) - U_B(θ = 0°) = mB −(-mB) = 2mB

Q) A bar magnet is suspended horizontally by a torsion less wire in magnetic meridian. In order to deflect the magnet through 30^o from the magnetic meridian, the upper end of the wire has to be rotated by 270^o. Now this magnet is replaced by another magnet. In order to deflect the second magnet through the same angle from the magnetic meridian, the upper end of the wire has to be rotated by 180^o. What is the ratio of the magnetic moments of the two bar magnets.

Solution

Let C be the deflecting torque per unit twist and M_1 and M_2 be the magnetic moments of the two magnets.

The deflecting torque is $\tau = C\theta$

The restoring torque is $\tau = MB \sin \theta$

In equilibrium,

deflecting torque = restoring torque

For the Magnet – I

$C(270^o - 30^o) = M_1 B_H \sin \theta$ ... (1)

For the magnet – II

$C(180^o - 30^o) = M_2 B_H \sin \theta$ ... (2)

Dividing (1) by (2)

$$\frac{M_1}{M_2} = \frac{240^o}{150^o} = \frac{8}{5}$$

Q) A magnetic needle placed in uniform magnetic field has magnetic moment 6.7×10^{-2} Am^2 and moment of inertia of 15×10^{-6} kgm^2. It performs 10 complete oscillations in 6.7 s. What is the magnitude of the magnetic field.

Solution:

The periodic time of oscillation is

$$T = 2\pi\sqrt{\frac{I}{mB}}$$

$$B = 4\pi^2 \frac{I}{mT^2}$$

$$B = \frac{4\pi^2(3.13)^2 \times 15 \times 10^{-6}}{6.7 \times 10^2 \times (0.67)^2} = 0.02T$$

Gauss's Law for Magnetic Field

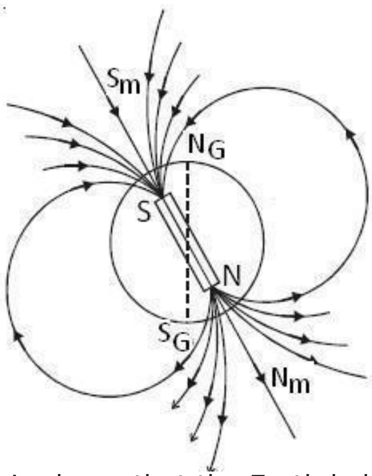
Magnetic field lines always forms a closed loops, the magnetic flux associated with any closed surface is always zero

$$\oint_{\substack{Closed\\Sufrace}} \vec{B} \cdot \vec{da} = 0$$

Where B is the magnetic field and ds is an infinitesimal area vector of the closed surface "The net magnetic flux passing through any closed surface is zero" This statement is called Gauss's law for magnetic field.

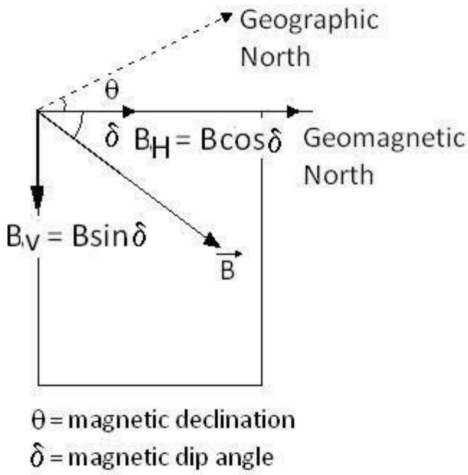
Earth's magnetic field and magnetic elements

A freely suspended magnetic needle at a point on Earth comes to rest approximately along the geographical north - south direction.



This shows that the Earth behaves like a huge magnetic dipole with its magnetic poles near its geographical poles. Since the north pole of the magnetic needle approximately points towards geographic north (NG) it is appropriate to call the magnetic pole near NG as the magnetic south pole of Earth Sm. Also, the pole near SG is the magnetic north pole of the Earth (Nm).
The Earth's magnetic field at any point on the Earth can be completely defined in terms of certain quantities called magnetic elements of the Earth, namely

(i) Declination or the magnetic variation θ.
The angle between the magnetic meridian and geographic meridian at a place on the surface of the earth is called magnetic declination at that place



(ii) Dip or inclination δ
Magnetic dip or angle of inclination is the angle δ (up or down) that the magnetic field of earth makes with the horizontal at a place in magnetic meridian
(iii) The horizontal and vertical component of the Earth's magnetic field.
B_V = B sinδ and B_H = Bcosδ

$$tan\delta = \frac{B_V}{B_H}$$

$$B = \sqrt{B_V^2 + B_H^2}$$

Causes of the Earth's magnetism
The exact cause of the Earth's magnetism is not known even today. However, some important factors which may be the cause of Earth's magnetism are:

(i) Magnetic masses in the Earth.
(ii) Electric currents in the Earth.
(iii) Electric currents in the upper regions of the atmosphere.
(iv) Radiations from the Sun.
(v) Action of moon etc.

However, it is believed that the Earth's magnetic field is due to the molten charged metallic fluid inside the Earth's surface with a core of radius about 3500 km compared to the Earth's radius of 6400 km.

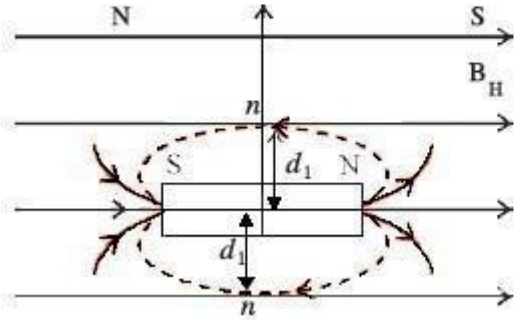
Solved Numerical

Q) A short bar magnet is placed with its north pole pointing north. The neutral point is 10 cm away from the centre of the magnet. If B = 4 × 10^{-5} T, calculate the magnetic moment of the magnet.

Solution:

When we keep North pole pointing north pole it means, it is in the direction of field lines of earth is opposite to magnetic field lines of magnet.

As shown in figure let neutral point (where effective magnetic field becomes zero) be at point n, at distance d_1 = 20 cm



Now magnetic field due to bar magnet = Horizontal component of earth

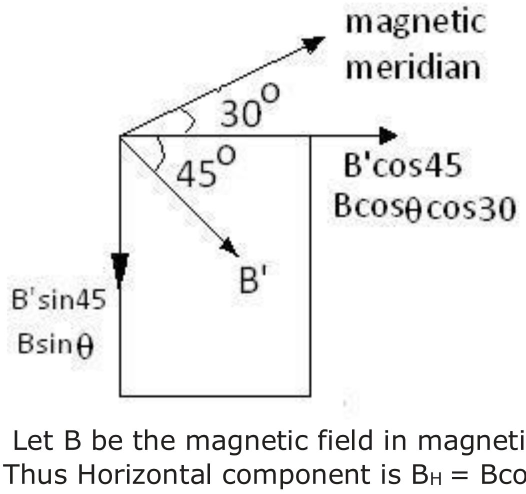
$$\frac{\mu_0}{4\pi}\frac{m}{d_1^3} = B_H$$

$$10^{-7}\frac{m}{(0.1)^3} = 4 \times 10^5$$

m=0.4 A m^2

Q) A magnet makes an angle of 45° with the horizontal in a plane making an angle of 30° with the magnetic meridian. Find the true value of the dip angle at the place.

Solution:



Let B be the magnetic field in magnetic meridian, making an angle of θ with horizontal.

Thus Horizontal component is B_H = Bcos θ and

vertical component is B_V = Bsinθ

Component of Horizontal component of magnetic field in magnetic meridian along plane = Bcosθcos30

Let magnetic field in plane be B'. Thus Horizontal component B'_H = B'cos45 and Vertical component B'_V = B'sin45

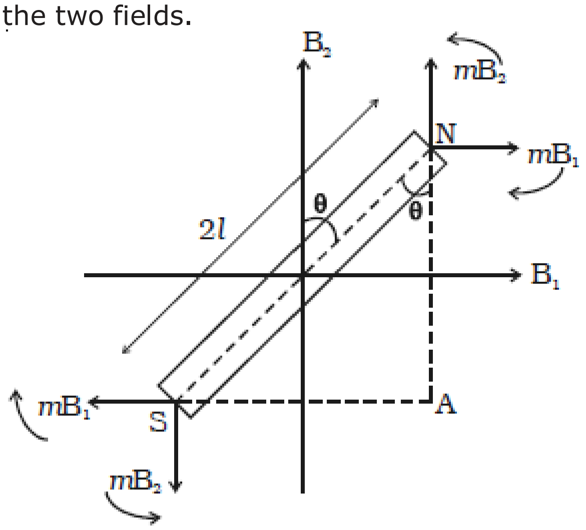
From above

Bsinθ = B'sin45 eq(1)
And
Bcosθcos30 =B'cos45 eq(2)
Taking ratio of eq(1) and eq(2) we get

$$\frac{B\sin\theta}{B\cos\theta\cos 30} = \frac{B'\sin 45}{B'\cos 45}$$
$$\tan\theta = \cos 30$$
$$\tan\theta = \frac{\sqrt{3}}{2} = 0.866$$
$$\theta = \tan^{-1}(0.866)$$

Tangent law

A magnetic needle suspended, at a point where there are two crossed magnetic fields acting at right angles to each other, will come to rest in the direction of the resultant of the two fields.



B_1 and B_2 are two uniform magnetic fields acting at right angles to each other. A magnetic needle placed in these two fields will be subjected to two torques tending to rotate the magnet in opposite directions. The torque τ_1 due to the two equal and opposite parallel forces mB_1 and mB_1 tend to set the magnet parallel to B_1. Similarly the torque τ_2 due to the two equal and opposite parallel forces mB_2 and mB_2 tends to set the magnet parallel to B_2. In a position where the torques balances each other, the magnet comes to rest. Now the magnet makes an angle θ with B_2 as shown in the Fig.

The deflecting torque due to the forces mB_1 and mB_1

$\tau_1 = mB_1 \times NA$
$= mB_1 \times NS \cos\theta$
$= mB_1 \times 2l \cos\theta$
$= 2l\, mB_1 \cos\theta$
$\therefore \tau_1 = MB_1\cos\theta$

Similarly the restoring torque due to the forces mB_2 and mB_2

$\tau_2 = mB_2 \times SA$
$= mB_2 \times 2l \sin\theta$
$= 2lm \times B2 \sin\theta$
$\tau_2 = MB2 \sin\theta$

At equilibrium,

$\tau_1 = \tau_2$
$\therefore MB_1 \cos\theta = MB_2 \sin\theta$
$\therefore B_1 = B_2 \tan\theta$

This is called Tangent law

Invariably, in the applications of tangent law, the restoring magnetic field B_2 is the horizontal component of Earth's magnetic field B_H.

Solved Numerical

A short bar magnet of magnetic moment 5.25×10^{-2} A m^2 is placed with its axis perpendicular to the Earth's field direction. At what distance from the centre of the magnet on (i) its equatorial line and (ii) its axial line, is the resultant field inclined at 45° with the Earth's field. Magnitude of the Earth's field at the place is 0.42×10^{-4} T.

Solution

From Tangent Law

$$\frac{B}{B_H} = tan\theta$$

$$B = B_H \tan\theta = 0.42 \times 10^{-4} \times \tan 45^o$$

$$B = 0.42 \times 10^{-4} \text{ T}$$

(i) For the point on the equatorial line

$$B = \frac{\mu_0}{4\pi}\frac{m}{d^3}$$

$$d^3 = \frac{\mu_0}{4\pi}\frac{m}{B}$$

$$d^3 = 10^{-7} \times \frac{5.25 \times 10^{-2}}{0.42 \times 10^{-4}}$$

$$d = 5\times10^{-2} \text{ m}$$

(ii) For the point on the axial line

$$B = \frac{\mu_0}{4\pi}\frac{2m}{d^3}$$

$$d^3 = \frac{\mu_0}{4\pi}\frac{2m}{B}$$

$$d^3 = 10^{-7} \times \frac{2 \times 5.25 \times 10^{-2}}{0.42 \times 10^{-4}}$$

$$d = 6.3 \times 10^{-2} \text{ m.}$$

Magnetic properties of materials

The study of magnetic properties of materials assumes significance since these properties decide whether the material is suitable for permanent magnets or electromagnets or cores of transformers etc.

Before classifying the materials depending on their magnetic behavior, the following important terms are defined.

(i) Magnetizing field or magnetic intensity

The magnetic field used to magnetize a material is called the magnetizing field. It is denoted by H and its unit is A m^{-1}.

(Note : Since the origin of magnetism is linked to the current, the magnetizing field is usually defined in terms of ampere turn)

(ii) Magnetic permeability

Magnetic permeability is the ability of the material to allow the passage of magnetic lines of force through it. *Relative permeability μ_r of a material is defined as the ratio of number of magnetic lines of force per unit area B inside the material to the number of lines of force per unit area in vacuum B_o produced by the same magnetizing field.*

∴ Relative permeability $\mu_r = B / B_O$

$$\mu_r = \frac{\mu H}{\mu_0 H} = \frac{\mu}{\mu_0}$$

(since μ_r is the ratio of two identical quantities, it has no unit.)

∴ The magnetic permeability of the medium $\mu = \mu_o\mu_r$ where μ_o is the

permeability of free space.
Magnetic permeability μ of a medium is also defined as the ratio of magnetic induction B inside the medium to the magnetizing field H inside the same medium.

$$\mu = \frac{B}{H}$$

(iii) Intensity of magnetization
Intensity of magnetization represents the extent to which a material has been magnetized under the influence of magnetizing field H. Intensity of magnetization of a magnetic material is defined as the magnetic moment per unit volume of the material.

$$M = \frac{m}{V}$$

Its unit is A m^{-1}.
For a specimen of length 2l, area A and pole strength m,

$$M = \frac{2lm}{2lA}$$

$$M = \frac{m}{A}$$

Hence, intensity of magnetization (M) is also defined as the pole strength per unit area of the cross section of the material.

(iv) Magnetic induction
When a soft iron bar is placed in a uniform magnetizing field H, the magnetic induction inside the specimen B is equal to the sum of the magnetic induction B_o produced in vacuum due to the magnetizing field and the magnetic induction B_m due to the induced magnetization of the specimen.
$B = B_o + B_m$
But $B_o = \mu_o H$ and $B_m = \mu_o M$
$B = \mu_o H + \mu_o M$
$\therefore B = \mu_o (H + M)$

$$H = \frac{B}{\mu_o} - M$$

where **H** has the same dimensions as **M** and is measured in units of A m^{-1}.
Thus, the total magnetic field **B**

(v) Magnetic susceptibility
Magnetic susceptibility χ_m is a property which determines how easily and how strongly a specimen can be magnetized.
Susceptibility of a magnetic material is defined as the ratio of intensity of magnetization induced in the material to the magnetizing field H in which the material is placed.
Thus

$$\chi_m = \frac{M}{H}$$

Since *I* and *H* are of the same dimensions, χ_m has no unit and is dimensionless.

Relation between χ_m and μ_r

$$\chi_m = \frac{M}{H}$$

$$M = \chi_m H$$

We know $B = \mu_o (H + M)$

$$B = \mu_o (H + \chi_m H)$$

$$B = \mu_o H (1 + \chi_m)$$

If μ is the permeability, we know that $B = \mu H$.

$$\therefore \mu H = \mu_o H (1 + \chi_m)$$

$$\frac{\mu}{\mu_0} = (1 + \chi_m)$$

$$\mu_r = 1 + \chi_m$$

Solved Numerical

A bar magnet of mass 90 g has magnetic moment 3 A m^2. If the intensity of magnetization of the magnet is 2.7 × 10^5 A m^{-1}, find the density of the material of the magnet.

Solution

Intensity of magnetization, $M = \frac{m}{V}$

volume V =mass/ ρ

$$M = \frac{m\rho}{mass}$$

$$\rho = \frac{M \times mass}{m} = \frac{2.7 \times 10^5 \times 0.090}{3}$$

$$\rho = 8100 \frac{kg}{m^3}$$

Q) A magnetizing field of 50 A m^{-1} produces a magnetic field of induction 0.024 T in a bar of length 8 cm and area of cross section 1.5 cm^2. Calculate (i) the magnetic permeability (ii) the magnetic susceptibility

Solution

Permeability

$$\mu = \frac{B}{H} = \frac{2.4 \times 10^{-2}}{50} = 4.8 \times 10^{-4} Hm^{-1}$$

susceptibility

$$\chi_m = \frac{\mu}{\mu_0} - 1$$

$$\chi_m = \frac{4.8 \times 10^{-4}}{4\pi \times 10^{-7}} - 1 = 381.16$$

Q) A solenoid has a core of material with relative permeability of 400. The current passing through the wire of solenoid is 2A. If the number of turns per cm are 10, calculate the magnitude of

(a) H (b) B (c) χ_m (d) M_

Solution

Here μ_r = 400, I = 2A n = 10 turns/cm = 1000 turns/m

(a) Magnetic intensity H = nI = 1000 × 2 = 2000 Am^{-1}
(b) Magnetic field B = $\mu_o \mu_r H$ = 4π×10^{-7}×400×2000 = 1.0 T
(c) Magnetic susceptibility of the core material is
    $\chi_m = \mu_r - 1$ = 400 -1 = 399
(d) Magnetization
    M = $\chi_m H$ = 399 × 2000 = 8 ×10^5 A/m

Q) The region inside a current carrying toroidial winding is filled with tungsten of susceptibility 6.8×10^{-5}. What is the percentage increase in the magnetic field in the presence of the material with respect to the magnetic field without it?

Solution:

The magnetic field in the current carrying toroidial winding without tungsten is

$B_O = \mu_O H$

The magnetic field in the same current carrying toroidial winding with tungsten is

B = μH

$$\therefore \frac{B - B_o}{B_o} = \frac{\mu - \mu_o}{\mu_o}$$

But $\mu = \mu_o(1 + \chi_m)$

$$\frac{\mu}{\mu_o} = 1 + \chi_m$$

$$\chi_m = \frac{\mu - \mu_O}{\mu_0}$$

$$\therefore \frac{B - B_o}{B_o} = \chi_m$$

$$\therefore \frac{B - B_o}{B_o} \times 100 = \chi_m \times 100$$

$$\therefore \frac{B - B_o}{B_o} \times 100 = (6.8 \times 10^{-5}) \times 100$$

$$\therefore \frac{B - B_o}{B_o} \times 100 = (6.8 \times 10^{-3})\%$$

Classification of magnetic materials

On the basis of the behavior of materials in a magnetizing field, the materials are generally classified into three categories namely,
(i) Diamagnetic, (ii) Paramagnetic and (iii) Ferromagnetic

(i) Properties of diamagnetic substances

Diamagnetic substances are those in which the net magnetic moment of atoms is zero. The susceptibility has a low negative value.
(For example, for bismuth $\chi m = -0.00017$).
2. Susceptibility is independent of temperature.
3. The relative permeability is slightly less than one.
4. When placed in a non uniform magnetic field they have a tendency to move away from the field (i.e) from the stronger part to the weaker part of the field. They get magnetized in a direction opposite to the field.
5. When suspended freely in a uniform magnetic field, they set themselves perpendicular to the direction of the magnetic field
Examples : Bi, Sb, Cu, Au, Hg, H_2O, H_2 etc.

(ii) Properties of paramagnetic substances

Paramagnetic substances are those in which each atom or molecule has a net non-zero magnetic moment of its own.
1. Susceptibility has a low positive value.
(For example : χ_m for aluminium is $+0.00002$).
2. Susceptibility is inversely proportional to absolute temperature. As the temperature increases susceptibility decreases.
3. The relative permeability is greater than one.
4. When placed in a non uniform magnetic field, they have a tendency to move from weaker part to the stronger part of the field. They get magnetized in the direction of the field.
5. When suspended freely in a uniform magnetic field, they set themselves parallel to the direction of magnetic field
Examples : Al, Pt, Cr, O_2, Mn, $CuSO_4$ etc.

Pierre Curie observed the magnetization M of a paramagnetic material is directly proportional to the external magnetic filed B and inversely proportional to its absolute temperature T, called Curie's law

$$M = C\frac{B}{T}$$

Where C = Curie's constant

$$M = C\frac{B}{T}\frac{\mu_0}{\mu_0}$$

$$M = CH\frac{\mu_0}{T} (\because H = \frac{B}{\mu_0})$$

$$\frac{M}{H} = \chi_m = C\frac{\mu_0}{T}$$

(iii) Properties of ferromagnetic substances

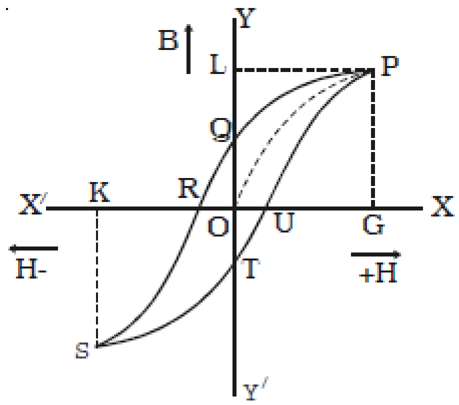
Ferromagnetic substances are those in which each atom or molecule has a strong spontaneous net magnetic moment. These substances exhibit strong paramagnetic properties.
1. The susceptibility and relative permeability are very large.
(For example : μ_r for iron = 200,000)
2. Susceptibility is inversely proportional to the absolute temperature.
As the temperature increases the value of susceptibility
decreases. *At a particular temperature, ferromagnetic become paramagnetic. This transition temperature is called Curie temperature.*
The relation between magnetic susceptibility of the substance in the acquired paramagnetic form and temperature is given by

$$\chi_m = \frac{C_1}{T - T_C}$$

C_1 is a constant
For example: Curie temperature of iron is about 1000 K.
3. When suspended freely in uniform magnetic field, they set themselves parallel to the direction of magnetic field.
4. When placed in a non uniform magnetic field, they have a tendency to move from the weaker part to the stronger part of the field. They get strongly magnetized in the direction of the field.
Examples : Fe, Ni, Co and a number of their alloys.

Hysteresis

Consider an iron bar being magnetized slowly by a magnetizing field H whose strength can be changed. It is found that the magnetic induction B inside the material increases with the strength of the magnetizing field and then attains a saturated level. This is depicted by the path OP in the



If the magnetizing field is now decreased slowly, then magnetic induction also decreases but it does not follow the path PO. Instead, when H = 0, B has non zero value equal to OQ. This implies that some magnetism is left in the specimen. *The value of magnetic induction of a substance, when the magnetizing field is reduced to zero, is called residual magnetic induction of the material.* OQ represents the residual magnetism of the material. Now, if we apply the magnetizing field in the reverse direction, the magnetic induction decreases along QR till it becomes zero at R. Thus to reduce the residual magnetism (remnant magnetism) to zero, we have to apply a magnetizing field OR in the opposite direction.
The value of the magnetizing field H which has to be applied to the magnetic material in the reverse direction so as to reduce its residual magnetism to zero is called its coercivity.
When the strength of the magnetizing field H is further increased in the reverse direction, the magnetic induction increases along RS till it acquires saturation at a point S (points P and S are symmetrical). If we now again change the direction of the field, the magnetic induction follows the path STUP. *This closed curve PQRSTUP is called the*

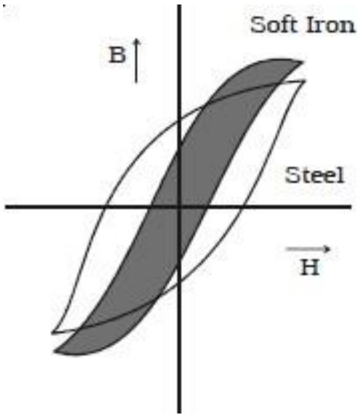
'hysteresis loop' and it represents a cycle of magnetization. The word 'hysteresis' literally means lagging behind. We have seen that magnetic induction B lags behind the magnetizing field H in a cycle of magnetization. *This phenomenon of lagging of magnetic induction behind the magnetizing field is called hysteresis.*

In the process of magnetization of a ferromagnetic substance through a cycle, there is expenditure of energy. The energy spent in magnetizing a specimen is not recoverable and there occurs a loss of energy in the form of heat. This is so because, during a cycle of magnetization, the molecular magnets in the specimen are oriented and reoriented a number of times. This molecular motion results in the production of heat. It has been found that *loss of heat energy per unit volume of the specimen in each cycle of magnetization is equal to the area of the hysteresis loop.* The shape and size of the hysteresis loop is characteristic of each material because of the differences in their retentivity, coercivity, permeability, susceptibility and energy losses etc. By studying hysteresis loops of various materials, one can select suitable materials for different purposes.

At H = 0, B ≠ 0. The value of B at H = 0 is called retentivity or remanence.

At H ≠ 0, B = 0. The value of H at B = 0 is called coercivity.

Hysteresis loss



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Uses of ferromagnetic materials

(i) Permanent magnets

The ideal material for making permanent magnets should possess high retentivity (residual magnetism) and high coercivity so that the magnetization lasts for a longer time. Examples of such substances are steel and alnico (an alloy of Al, Ni and Co).

(ii) Electromagnets

Material used for making an electromagnet has to undergo cyclic changes. Therefore, the ideal material for making an electromagnet has to be one which has the least hysteresis loss. Moreover, the material should attain high values of magnetic induction B at low values of magnetizing field H. Soft iron is preferred for making electromagnets as it has a thin hysteresis loop[small area, therefore less hysteresis loss] and low retentivity.It attains high values of B at low values of magnetizing field.

(iii) Core of the transformer

A material used for making transformer core and choke is subjected to cyclic changes very rapidly. Also, the material must have a large value of magnetic induction B. Therefore, soft iron that has thin and tall hysteresis loop is preferred. Some alloys with low hysteresis loss are: radio-metals, pern-alloy.

(iv) Magnetic tapes and memory store

Magnetization of a magnet depends not only on the magnetizing field but also on the cycle of magnetization it has undergone. Thus, the value of magnetization of the specimen is a record of the cycles of magnetization it has undergone. Therefore, such a system can act as a device for storing memory. Ferro magnetic materials are used for coating magnetic tapes in a cassette player and for building a memory store in a modern computer. Examples : Ferrites (Fe, Fe_2O, $MnFe_2O_4$ etc.).

Questions

Q) It is observed that the neutral points lie along the axis of a magnet placed on the table. What is the orientation of the magnet with respect to the earth's magnetic field

Ans. North pole of the magnet is towards the south of the earth

Q) A bar magnet is stationary in magnetic meridian. Another similar magnet is kept to it such that the centre lie on their perpendicular bisectors. If the second magnet is free to move, then what type of motion it will have - translator, rotator or both

Ans: Only translator

Q) A short bar magnet placed with its axis making an angle θ with a uniform external field B experiences a torque. What is the magnetic moment of the magnet

Q) Name the parameters needed to completely specify the earth's magnetic field at a point on the earth's surface

Ans: Declination, Dip and Horizontal component of earth's field

Q) What is geomagnetic equator

Ans: The great circle on the earth's surface whose plane is perpendicular to the magnetic axis is called magnetic equator.

Q) What is magnetic meridian

Ans: A vertical plane passing through the magnetic axis of earth is called magnetic meridian

Q) Name the physical quantity which is measured in Wb A^{-1}

Ans: The ratio of the magnetic induction and the magnetic moment is measured in Wb A^{-1}

Q) Name one property of magnetic material used for making permanent magnet

Ans: High coercivity

Q) The ratio of the horizontal component to the resultant magnetic field of earth at a given place is $(1/\sqrt{2})$. What is the angle of dip at that place

Ans : $\cos\theta = \frac{B_H}{B} = \frac{1}{\sqrt{2}}$

$\theta = 45^O$

Q) Why does a paramagnetic sample display greater magnetization (for same magnetizing field) when cooled

Ans: The tendency to disrupt the alignment of dipoles with the magnetizing field arising from random thermal motion is reduced at lower temperatures. So, as the paramagnetic substance is cooled, its atomic dipoles tends to get aligned with the magnetizing field. Thus, the paramagnetic substance display a greater magnetization when cooled

Q) What is SI unit of magnetic permeability?

Ans: T m A^{-1}

Q) Why do magnetic lines of force prefer to pass through iron than air

Ans: Permeability of soft iron is greater than that of air

Q) What is the SI unit of susceptibility

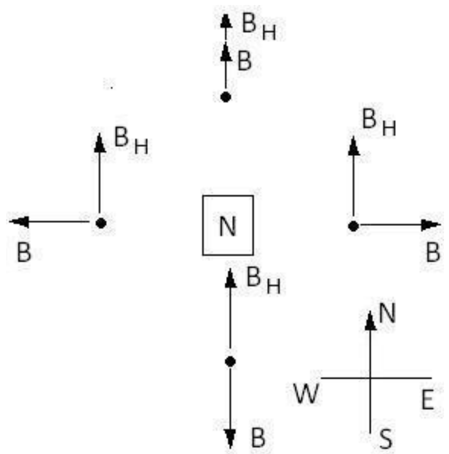
Ans: It has no unit

Q) Identify a substance, which has negative magnetic susceptibility.

Ans: Diamagnetic substance. Magnetic susceptibility is positive for both para and ferromagnetic substance

Q) What is the net magnetic moment of an atom of a diamagnetic material

Ans: Zero
Q) What is the dimensional formula of magnetic flux
Ans: [$ML^2T^{-2}A^{-1}$]
Q) An iron nail is attracted by a magnet. What is the source of kinetic energy
Ans: It is the magnetic field energy which is partly converted into kinetic energy
Q) A bar magnet is cut into two equal pieces transverse to its length. What happens to its dipole moment
Ans: The magnetic moment will be halved because length will be halved
Q) What is magnet
Ans: A magnet is an arrangement of two equal and opposite magnetic poles separated by a certain distance. It has attractive and directive properties
Q) What is the SI unit of magnetic moment of a dipole
Ans: Am^2 or JT^{-1}
Q) What is Hysteresis?
Ans: Hysteresis is defined as the lagging of the magnetic induction B behind the corresponding magnetic field H
Q) Define angle of magnetic dip
Ans: It is the angle made by the direction of earth's total magnetic field with the horizontal component of the earth's magnetic field at magnetic poles
Q) What is the effect on the magnetization of diamagnetic substance when it is cooled
Ans: The magnetization of a diamagnetic substance is independent of temperature
Q) A magnet is held vertically on a horizontal plane. How many neutral points are there in the horizontal plane



Ans.) The magnetic field due to the magnet and the magnetic field of earth are shown at four different points a, b, c and d. Clearly, the two fields cancel only at the point a. So, a is the neutral point.

Q) In the stirrup of a vibration magnetometer are placed two magnets one above the other with their axes parallel. When will their time period be maximum/minimum
Ans: The time period will be maximum when opposite poles are together

$$T_{max} = 2\pi\sqrt{\frac{I_1 + I_2}{(m_1 - m_2)B_H}}$$

The time period will be minimum when like poles are together

$$T_{max} = 2\pi\sqrt{\frac{I_1 + I_2}{(m_1 + m_2)B_H}}$$

Q) Two substances A and B have their relative permeability slightly greater and less than unity respectively. What do you conclude about A and B
Ans: $\chi_m = \mu_r - 1$
Relative permeability of A is slightly greater than 1. So χ_m is small and positive. So, substance is paramagnetic.
Relative permeability of B is slightly less 1
So χ_m is small and negative. Clearly, substance is diamagnetic
Q) How does the knowledge of declination at a place help in navigation?

Ans: Declination at place gives us the angle between the geographic and the magnetic meridians. So, the knowledge of declination shall help in steering the ship in the required direction so as to reach the destination

Q) Two identical-looking iron bars A and B given, one of which is definitely known to be magnetized [We don't know which one]. How would one ascertain whether or not both are magnetized? If only one is magnetized, how does one ascertain which one? [Use nothing but the two bars A and B]

Ans: Try to bring different ends of the magnets closer. A repulsive force in some situation establishes that both are magnetized. If it is always attractive, then one of them is not magnetized. To see which one, pick up one say A and lower one of its ends: first one of the ends of their other say b, and then on the middle part of B. A experiences no force, and then B is magnetized. If you do not notice any change from end to middle point Of B, then A is magnetized.

Q) A magnetized needle in a uniform magnetic field experiences a torque but no net force. An iron nail near a bar magnet, however, experiences a force of attraction in addition to a torque. Why?

Ans: In the case of uniform magnetic field, the forces experienced by the needle are equal in magnitude, opposite in direction and have different lines of action. So, net force is zero. But torque is not zero

The iron nail experiences a non-uniform magnetic field due to the bar magnet. The induced magnetic moment in the nail, therefore, experiences both force and torque. The net force is attractive because the induced (say) south pole in the nail is closer to the north pole of the magnet than the induced north pole

Q) Why two magnetic lines of force due to a bar magnet do not cross each other?

Ans: If two magnetic lines of force cross at a point, then this would mean that there are two directions of magnetic field at the point of crossing. This is physically absurd. Thus, two magnetic lines of force cannot cross each other

Q) What is the basic use of hysteresis curve?

Ans: Hysteresis loop gives useful information about the different properties, of materials, such as coercivity, retentivity, energy loss. This information helps us in the suitable selection of materials for different purposes.

Q) Does the magnetization of paramagnetic salt depend on temperature? Justify your answer

Ans: The atoms of a paramagnetic substance posses small magnetic dipole moments. But these atomic dipoles are oriented in a random manner. In the presence of the external magnetic field, these dipoles tend to align in the direction of the field. But the tendency for alignment is hindered by thermal agitation. So, the magnetization of paramagnetic salt decreases with increase of temperature.

Motion In One Dimension

Particle

A particle is ideally just a piece or a quantity of matter, having practically no linear dimensions but only a position.

In practice it is difficult to get such particle, but in certain circumstances an object can be treated as particle.

Such circumstances are

(i) All the particles of solid body performing linear motion cover the same distance in the same time. Hence motion of such a body can be described in terms of the motion of its constituent particle

(ii) If the distance between two objects is very large as compared to their dimensions, these objects can be treated as particles. For example, while calculating the gravitational force between Sun and Earth, both of them can be considered as particles.

Frame of reference

A "frame of reference" is just a set of coordinates: something you use to measure the things that matter in Newtonian problems, that is to say, positions and velocities, so we also need a clock.

Or A place and situation from where an observer takes his observation is called frame of reference.

A point in space is specified by its three coordinates (x, y, z) and an "event" like, say, a little explosion, by a place and time: (x, y, z, t).

An inertial frame is defined as one in which Newton's law of inertia holds—that is, anybody which isn't being acted on by an outside force stays at rest if it is initially at rest, or continues to move at a constant velocity if that's what it was doing to begin with. Example of inertial frame of reference is observer on Earth for all motion on surface of earth. Car moving with constant velocity

An example of a non-inertial frame is a rotating frame, such as a accelerating car,

Rest and Motion

When a body does not change its position with respect to time with respect to frame of reference, then it is said to be at rest. Revolving earth *Motion is the change of position of an object with respect to time.*

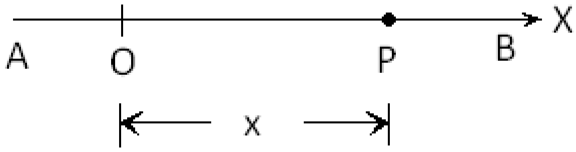
To study the motion of the object, one has to study the change in position (*x,y,z* coordinates) of the object with respect to the surroundings. It may be noted that the position of the object changes even due to the change in one, two or all the three coordinates of the position of the objects with respect to time. Thus motion can be classified into three types:

(i) Motion in one dimension

Motion of an object is said to be one dimensional, if only one of the three coordinates specifying the position of the object changes with respect to time.

Example : An ant moving in a straight line, running athlete, etc.

Consider a particle moving on a straight line AB. For the analysis of motion we take origin. O at any point on the line and x-axis along the line. Generally we take origin at the point from where particle starts its motion and rightward direction as positive x-direction. At any moment if article is at P then its position is given by OP = x



(ii) Motion in two dimensions

In this type, the motion is represented by any two of the three coordinates. Example: a body moving in a plane.

(iii) Motion in three dimensions

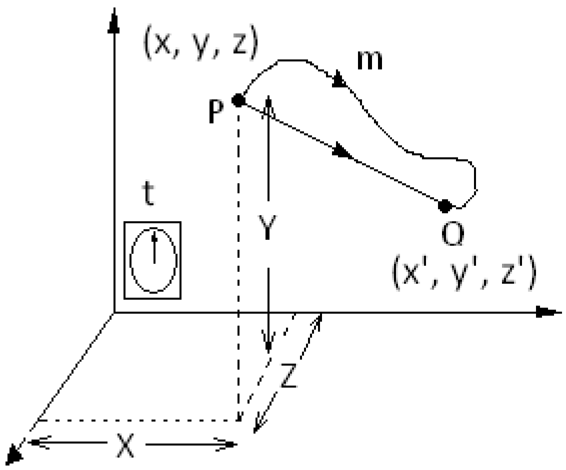
Motion of a body is said to be three dimensional, if all the three coordinates of the position of the body change with respect to time.

Examples : motion of a flying bird, motion of a kite in the sky, motion of a molecule, etc

Position, Path-length and Displacement

POSITION

Choose a rectangular coordinate system consisting of three mutually perpendicular axes, labeled X-, Y-, and Z- axes. The point of intersection of these three axes is called origin (O) and serves as the reference point, the coordinates (x,y,x) of a particle at point P describe the position of the object with respect to this frame of reference. To measure the time we put clock in this system



If all the coordinate of particle remains unchanged with time then particle is considered at rest with respect to this frame of reference.

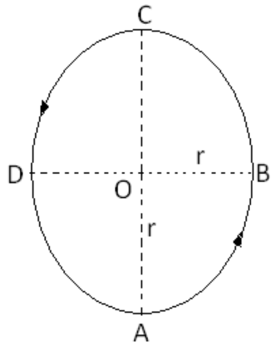
If position of particle at point P given by coordinates (x, y, z) at time t and particles position coordinates are (x', y', z') at time t' , that is at least one coordinates of the particle is changed with time then particle is said to be in motion with respect to this frame of reference

PATH LENGTH
The path length of an object in motion in a given time is the length of actual path traversed by the object in the given time. As shown in figure actual path travelled by the particle is PmO . Path length is always positive

DISPLACEMENT
The displacement of an object in motion in a given time is defined as the change in a position of the object, i.e., the difference between the final and initial positions of the object in a given time. It is the shortest distance between the two positions of the object and its directions is from initial to final position of the object, during the given interval of time. It is represented by the vector drawn from the initial position to its final position. As shown in figure. Since displacement is vector it may be zero, or negative also

Solved numerical

Q) A particle moves along a circle of radius r. It starts from A and moves in anticlockwise direction as shown in figure. Calculate the distance travelled by the particle and magnitude of displacement from each of following cases
(i) from A to B (ii) from A to C (iii) from A to D (iv) one complete revolution of the particle



Solution
(i) Distance travelled by particle from A to B is One fourth of circumference thus

$$path\ length = \frac{2\pi r}{4} = \frac{\pi r}{2}$$

Displacement

$$|AB| = \sqrt{(OA)^2 + (OB)^2} = \sqrt{r^2 + r^2} = \sqrt{2}\,r$$

(ii) Distance travelled by the particle from A to C is half of the circumference

$$path\ length = \frac{2\pi r}{2} = \pi r$$

Displacement
|AC| = r+r = 2r
(iii) Distance travelled by the particle from A to D is three fourth of the circumference

$$path\ length\ = 2\pi r\ \frac{3}{4} = \frac{3}{2}\pi r$$

Displacement AD

$$|AD| = \sqrt{(OA)^2 + (OD)^2} = \sqrt{r^2 + r^2} = \sqrt{2}\ r$$

(iv) For one complete revolution total distance is equal to circumference of circle

Path length = 2πr

Since initial position and final position is same displacement is zero

Speed and velocity

Speed

It is the distance travelled in unit time. It is a scalar quantity.

$$speed = \frac{path\ length}{time}$$

Solved numerical

Q) A motorcyclist covers 1/3rd of a given distance with speed 10 kmh^{-1}, the next 1/3rd at 20 kmh^{-1} and the last 1/3rd at of 30kmh^{-1}. What is the average speed of the motorcycle for the entire journey

Solution:

Let total distance or path length be 3x

Time taken for first 1/3rd path length

$$t_1\ = \frac{path\ length}{speed} = \frac{x}{10}\ hr$$

Time taken for second 1/3rd path length

$$t_1\ = \frac{path\ length}{speed} = \frac{x}{20}\ hr$$

Time taken for third 1/3rd path length

$$t_1\ = \frac{patn\ length}{speed} = \frac{x}{30}\ hr$$

Total time taken to travel path length of 3x is, $t = t_1+t_2+t_3$

Substituting values of t_1 t_2 and t_3 in above equation we get

$$t = \frac{x}{10} + \frac{x}{20} + \frac{x}{30} = \frac{11x}{60}\ hr$$

Form the formula for speed

$$speed\ = \frac{path\ length}{time}$$

$$speed\ = \frac{path\ length}{time}$$

$$speed\ = \frac{3x}{\frac{11x}{60}} = \frac{180}{11} = 16.36\ kmh^{-1}$$

Velocity

The velocity of a particle is defined as the rate of change of displacement of the particle. It is also defined as the speed of the particle in a given direction. The velocity is a vector quantity. It has both magnitude and direction.

$$velocity = \frac{displacement}{time}$$

Units for velocity and speed is m s^{-1} and its dimensional formula is LT^{-1}.

Uniform velocity

A particle is said to move with uniform velocity if it moves along a fixed direction and covers equal displacements in equal intervals of time, however small these intervals of time maybe.

Non uniform or variable velocity

The velocity is variable (non-uniform), if it covers unequal displacements in equal intervals of time or if the direction of motion changes or if both the rate of motion and the direction change.

Average velocity

Let s_1 be the position of a body in time t_1 and s_2 be its position in time t_2 The average velocity during the time interval ($t_2 - t_1$) is defined as

$$v = \frac{s_2 - s_1}{t_2 - t_1} = \frac{\Delta s}{\Delta t}$$

Average speed of an object can be zero ,positive or zero. It depends on sign of displacement.

In general average speed of an object can be equal to or greater than the magnitude of the average velocity

Instantaneous velocity

It is the velocity at any given instant of time or at any given point of its path. The instantaneous velocity *v* is given by

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

Acceleration

If the magnitude or the direction or both of the velocity changes with respect to time, the particle is said to be under acceleration. Acceleration of a particle is defined as the rate of change of velocity.

If object is performing circular motion with constant speed then also it is accelerated motion as direction of velocity is changing

Acceleration is a vector quantity.

$$acceleration = \frac{change\ in\ velocity}{time}$$

If *u* is the initial velocity and *v,* the final velocity of the particle after a time *t*, then the acceleration,

$$a = \frac{v-u}{t}$$

Its unit is m s^{-2} and its dimensional formula is LT^{-2}

The instantaneous acceleration is

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}$$

If the velocity decreases with time, the acceleration is negative. The negative acceleration is called **retardation or deceleration**

Equations of motion

Motion in straight line with uniform velocity

If motion takes place with uniform velocity v on straight line the

Displacement in time t , S = vt------- eq(1)

Acceleration of particle is zero

Motion in a straight line with uniform acceleration – equations of motion

Let particle moving in a straight line with velocity u (velocity at time t = 00 and with uniform acceleration a. Let its velocity be v at the end of the interval of time t (final velocity at time t). Let S be the displacement at the instant t acceleration a is

$$a = \frac{v-u}{t} \quad or$$

$$v = u + at \quad ---eq(2)$$

If u and a are in same direction 'a" is positive and hence final velocity v will be more than initial velocity u, velocity increases

If u and a are in opposite direction final velocity v will be less than initial velocity u. Velocity is decreasing. And acceleration is negative

Displacement during time interval t = average velocity ×t

$$S = \frac{v+u}{2} \times t - -eq(3)$$

Eliminating v from equation 3 and equation 2 we get

$$S = \frac{u + at + u}{2} \times t$$

$$S = ut + \frac{1}{2}at^2 \quad ---eq(4)$$

Another equation can be obtained by eliminating t from equation 2 and equation3

$$v = u + at$$

$$t = \frac{v-u}{a}$$

$$S = \frac{v+u}{2} \times \frac{v-u}{a}$$

$$S = \frac{v^2 - u^2}{2a}$$

$$v^2 = u^2 + 2aS - - - eq(5)$$

Distance transverse by the particle in n[th] second of its motion
The velocity at the beginning of the nth second = u +a (n-1)
The velocity at the end of n[th] second = u+an
Average velocity during n[th] second v_{ave}

$$v_{ave} = \frac{\mathrm{u + a\,(n-1) + u + an}}{2}$$

$$v_{ave} = u + \frac{1}{2}a\,(2n-1)$$

Distance during this one second
S_n = average velocity × time

$$S_n = u + \frac{1}{2}a(2n-1) \times 1$$

$$S_n = u + \frac{1}{2}a\,(2n-1) - - - eq(6)$$

The six equations derived above are very important and are very useful in solving problems in straight-line motion

Calculus method of deriving equation of motion
The acceleration of a body is defined as

$$a = \frac{dv}{dt}$$

$$dv = adt$$

Integrating we get v = at+A
Where A is constant of integration . For initial condition t = 0, v = u (initial velocity) we get A = u
∴ v = u +at

We know that instantaneous velocity v

$$v = \frac{ds}{dt}$$

ds = adt
displacement ds = vdt = (u+at)dt
integrating above equation

$$S = ut + \frac{1}{2}at^2 + B$$

B is integration constant
At t = 0, S = 0 yields B = 0

$$\therefore S = ut + \frac{1}{2}at^2$$

Acceleration a

$$a = \frac{dv}{dt} = \frac{dv}{dS} \cdot \frac{dS}{dt} = v\,\frac{dv}{dS}$$

$$\therefore a = v\frac{dv}{dS}$$

$$adS = v \cdot dv$$

Integrating we get

$$aS = \frac{v^2}{2} + C$$

Where C is integration constant

Applying initial condition , where S = 0, v = u we get

$$0 = \frac{u^2}{2} + C$$

$$Or\ \ C = -\frac{u^2}{2}$$

$$\therefore aS = \frac{v^2}{2} - \frac{u^2}{2}$$

$$v^2 = u^2 + 2aS$$

If S_1 and S_2 are the distances traversed during n seconds and (n-1) seconds

$$S_1 = un + \frac{1}{2}an^2$$

$$S_2 = u(n-1) + \frac{1}{2}a(n-1)^2$$

Displacement in n[th] second

$$S_n = S_1 - S_2$$

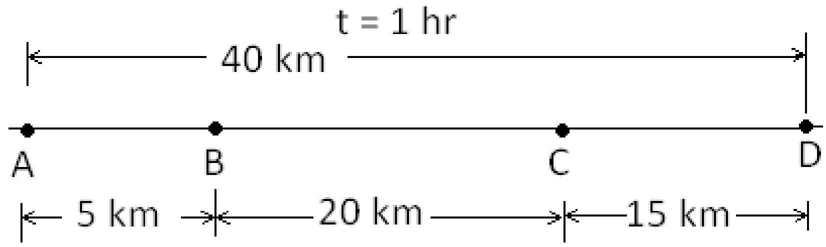
$$S_n = un + \frac{1}{2}an^2 - u(n-1) - \frac{1}{2}a(n-1)^2$$

$$S_n = u + \frac{1}{2}a(2n-1)$$

Solved numerical

Q) The distance between two stations is 40 km. A train takes 1 hour to travel this distance. The train, after starting from the first station, moves with constant acceleration for 5km, then it moves with constant velocity for 20 km and finally its velocity keeps on decreasing continuously for 15 km and it stops at the other station. Find the maximum velocity of the train.

Solution:



Motion is divided in three parts
Motion between point A and B is with constant acceleration
Here initial velocity u = 0 and final velocity at point B = v_{max}
Let time interval be t_1
From equation

$$S = \frac{v+u}{2} \times t$$
$$5 = \frac{v_{max} + 0}{2} \times t_1$$
$$t_1 = \frac{10}{v_{max}}$$

Motion between point B and C is with constant velocity v_{max}
Let time period t_2
Form formula S = vt
20 = v_{max} t_2

$$t_2 = \frac{20}{v_{max}}$$

Motion between point C and D is with retardation
Initial velocity is v_{max} and final velocity v= 0 let time interval t_3
From formula

$$S = \frac{v+u}{2} \times t$$
$$15 = \frac{0 + v_{max}}{2} \times t_3$$

$$t_3 = \frac{30}{v_{max}}$$

Total time taken is 1 hr
T = $t_1 + t_2 + t_3$

$$1 = \frac{10}{v_{max}} + \frac{20}{v_{max}} + \frac{30}{v_{max}}$$

$$\therefore V_{max} = 60 \text{ km h}^{-1}$$

Q) A certain automobile manufacturer claims that its sports car will accelerate from rest to a speed of 42.0 m/s in 8.0 s. under the important assumption that the acceleration is constant
(i) Determine the acceleration
(ii) Find the distance the car travels in 8s
(iii)Find the distance travelled in 8^{th} s

Solution

(a) Here initial velocity u = 0 and final velocity v = 42 m/s

From formula

$$a = \frac{v-u}{t}$$

$$a = \frac{42-0}{8} = 5.25\ ms^{-2}$$

(b) Distance travelled in 8.0s

From formula

$$S = ut + \frac{1}{2}at^2$$

$$S = (0)(t) + \frac{1}{2}(5.25)(8)^2 = 168\ m$$

(c) distance travelled in 8^{th} second.

From formula

$$S_n = u + \frac{1}{2}a\,(2n-1)$$

$$S_n = 0 + \frac{1}{2}(5.25)(2\times 8-1) = 39.375\ m$$

Q) Motion of a body along a straight line is described by the equation $x = t^3 + 4t^2 - 2t + 5$ where x is in meter and t in seconds

(a) Find the velocity and acceleration of the body at t = 4s

(b) Find the average velocity and average acceleration during the time interval from t = 0 to t = 4 s

Solution

(a)We have to find instantaneous velocity at t = 4s

$$v = \frac{dx}{dt} = \frac{d}{dt}(t^3 + 4t^2 - 2t + 5)$$

$$v = \frac{d}{dt}t^3 + 4\frac{d}{dt}t^2 - 2\frac{d}{dt}t + \frac{d}{dt}5$$

$$v = 3t^2 + 4\times 2t - 2$$

$$v = 3t^2 + 8t - 2$$

Thus we get equation for velocity, by substituting t = 4 in above equation we get instantaneous velocity at t = 4

$v=3(4)^2 + 8(4) - 2$

v = 78 m/s

To find instantaneous acceleration at t = 4s

$$a = \frac{dv}{dt} = \frac{d}{dt}(3t^2 + 8t - 2)$$

a =6t +8

Thus we get equation for acceleration, by substituting t=4 in equation for acceleration we get instantaneous acceleration t=4

a = 6(4) + 8

a=32 m s^{-2}

(b)Average velocity

Final position of object at time t = 4 s

$X_4 = (4)^3 + 4(4)^2 - 2(4) + 5 = 125$

Initial position of object at time t = 0 s

$X_0 = (0)^3 + 4(0)^2 - 2(0) + 5 = 5$

Displacement = 125 – 5 = 120 m, time interval t =4 seconds

Average velocity = Displacement / time = 120/4 = 30 ms^{-1}

Average acceleration

Initial velocity t = 0 from equation for velocity

$$v = 3t^2 + 8t - 2$$

$$v = 3(0)^2 + 8(0) - 2 = -2\ ms^{-1}$$

∴ Initial velocity u = -2 ms^{-1}

Final velocity is calculated as 78 ms^{-1}

From formula for average acceleration

$$a = \frac{v-u}{t} = \frac{78-(-2)}{4} = 20ms^{-2}$$

Q) A particle moving in a straight line has an acceleration of (3t – 4) ms^{-2} at time t seconds. The particle is initially 1m from O, a fixed point on the line, with a velocity of 2ms^{-1} . Find the time when the velocity is zero. Find the displacement of particle from O when t = 3

Solution:

$$a = \frac{dv}{dt}$$

$$\frac{dv}{dt} = 3t - 4$$

$$\Rightarrow \int_2^v dv = \int_0^t (3t-4)dt$$

$$\Rightarrow v - 2 = \frac{3t^2}{2} - 4t$$

$$\Rightarrow v = \frac{3t^2}{2} - 4t + 2$$

The velocity will be zero when

$$\frac{3t^2}{2} - 4t + 2 = 0$$

when

$$(3t-2)(t-2) = 0$$

$$t = \frac{2}{3}\ or\ 2$$

Using

$$\frac{ds}{dt} = v$$

We have

$$\frac{ds}{dt} = \frac{3t^2}{2} - 4t + 2$$

$$\Longrightarrow \int_1^s ds = \int_0^3 \left(\frac{3t^2}{2} - 4t + 2\right) dt$$

$$\Longrightarrow s - 1 = \left[\frac{3t^2}{2} - 4t + 2\right]_0^2 = 1.5$$

$$\Longrightarrow s = 2.5 \ m$$

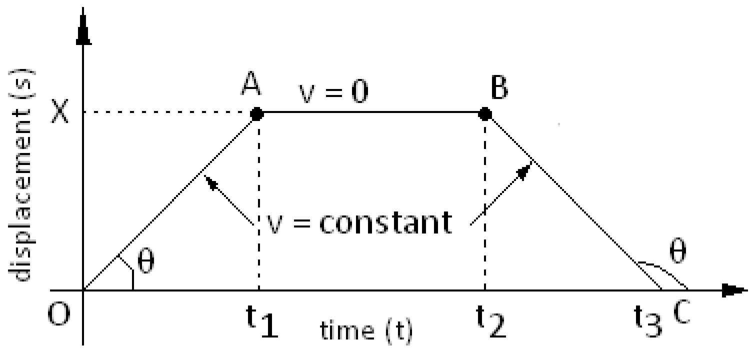
Therefore the particle is 2.5 m from O when t = 3s

Graphical representation of motion

(1) Displacement – time graph:

If displacement of a body is plotted on Y-axis and time on X-axis, the curve obtained is called displacement-time graph.

The instantaneous velocity at any given instant can be obtained from the graph by finding the slop of the tangent at the point corresponding to the time

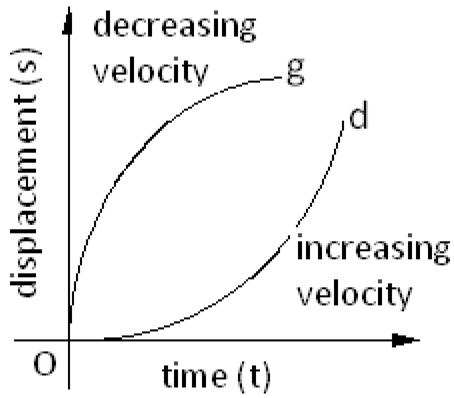


graph(A)

In graph(A) object started to move with constant velocity (a = 0) at time t = 0 from origin. Object is going away represented by OA, at time t_1 object reach position X ,note slope of graph AO is positive and constant .

For time period t_1 to t_2 object have not changed its position thus velocity is zero. Slope of graph is zero

For time period t_2 to t_3 object started to move towards its original position at time t_2 and reaches original position at time t_3 . Here velocity is constant (a=0) as slope of graph is constant. And reaching original position as slope is negative



graph (B)

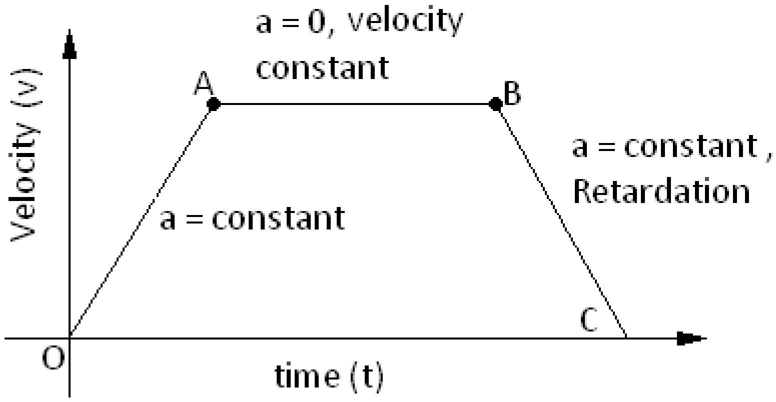
In graph(B) motion represented by Og is decelerated motion as slope is decreasing with time, hence velocity is decreasing. However object is moving away from origin

Motion represented by Od is accelerated as slope is continuously increasing with time, it indicates that velocity is increasing or acceleration is positive , object is moving away from origin

(2)Velocity-time graph

If Velocity of a body is plotted on Y-axis and time on X-axis, the curve obtained is called velocity-time graph.

The instantaneous acceleration at any given instant can be obtained from the graph by finding the slop of the tangent at the point corresponding to the time

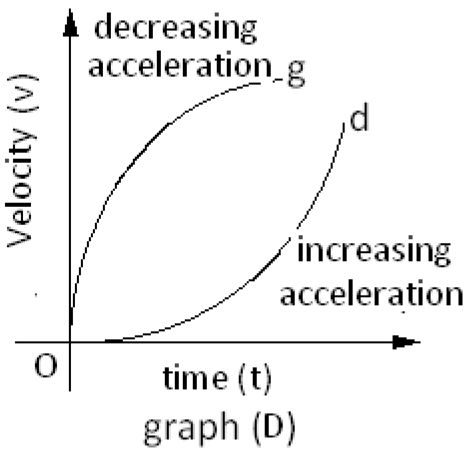


graph (C)

Graph AB is parallel straight indicate object is moving with constant velocity or acceleration is zero

Graph OA is oblique straight line slope is positive indicate object is uniformly accelerated

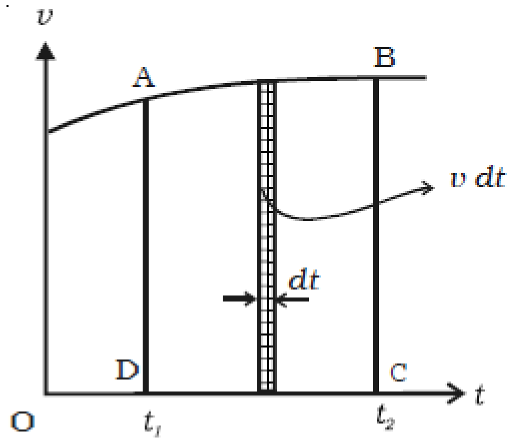
Graph BC is oblique straight line slope is negative indicated object is uniformly decelerated



Graph Og represents decreasing acceleration as slop is decreasing with time

Graph Od represent increasing acceleration as slop is increasing with time

When the velocity of the particle is plotted as a function of time, it is velocity-time graph. Area under the curve gives displacement



We know that

$$v = \frac{dS}{dt}$$

dS=v·dt

If displacements are S_1 and S_2 at time t_1 and t_2 then

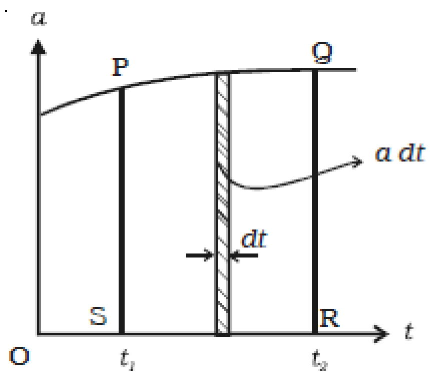
$$\int_{S_1}^{S_2} dS = \int_{t_1}^{t_2} v\,dt$$

$$S_2 - S_1 = \int_{t_1}^{t_2} v\,dt = Area\ ABCD$$

The area under the *v – t* curve, between the given intervals of time, gives the change in displacement or the distance travelled by the particle during the same interval.

Acceleration – time graph

When the acceleration is plotted as a function of time, it is acceleration - time graph



$a = \frac{dv}{dt}$

dv = a dt

If v_1 and v_2 are the velocities at time t_1 and t_2 then

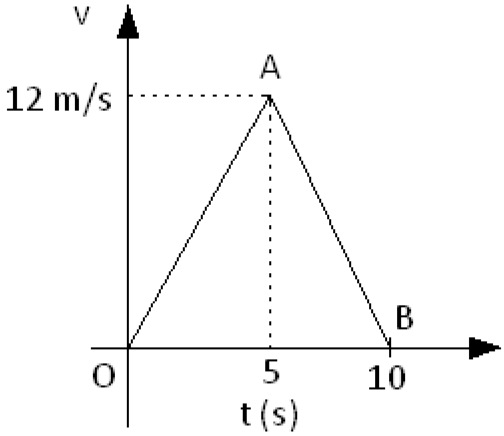
$$\int_{v_1}^{v_2} dy = \int_{t_1}^{t_2} adt$$

$$v_2 - v_1 = \int_{t_1}^{t_2} a\,dt = AreaPQRS$$

The area under the $a - t$ curve, between the given intervals of time, gives the change in velocity of the particle during the same interval. If the graph is parallel to the time axis, the body moves with constant acceleration.

Solved numerical

Q) The v – t graph of a particle moving in straight line is shown in figure. Obtain the distance travelled by the particle from (a) t = 0 to t = 10s and from (b) t = 2s to 6s



Solution:

(a) Distance travelled in time period t = 0 to t = 10s is area of triangle OAB = (1/2)×10×12 = 60 m

(b)Distance in time period t = 2 to t = 6s

From graph slope of line OA is 2.4 m/s^2

Initial velocity at t = 2 sec u = 4.8 thus using formula

$X = ut + (1/2)at^2$ here time period is 3 sec

$X_1 = (4.8)(3) + (1/2)(2.4)(3)^2 = 25.2$

For segment A to B acceleration is 2.4 time period 1s u = 5

$X_2 = (12)(1) – (1/2)(2.4)(1)^2 = 10.8$

Thus distance = 25.2 +10.8 = 36 m

Vertical motion under gravity

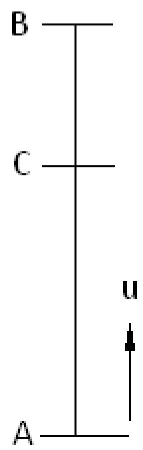
When an object is thrown vertically upward or dropped from height, it moves in a vertical straight line. If the air resistance offered by air to the motion of the object is neglected, all objects moving freely under gravity will be acted upon by its weight only

This causes vertical acceleration g having value 9.8 m/s^2, so the equation for motion in a straight line with constant acceleration can be used.

In some problems it is convenient to take the downward direction of acceleration as positive, in such case if the object is moving upward initial velocity should be taken as negative and displacement positive.

If object is moving downwards then, initial velocity should be taken as positive and displacement negative.

Projection of a body vertically upwards



Suppose an object is projected upwards from point A with velocity u

If we take down word direction of g as **Negative** then

(i) At a time t its velocity v = u –gt

(ii) At a time t, its displacement from A is gen by

$S = ut - (1/2) gt^2$

(iii) Its velocity when its displacement S is given by

$v^2 = u^2 - 2gS$

(iv) When it reaches the maximum height, its velocity v = 0.

This happens when t = u/g. The body is instantaneously rest

From formula

$$V = u - gt$$
$$t = v/g$$

(v) The maximum height reached. At maximum height final velocity v = 0 and S = H thus

From equation

$$v^2 = u^2 - 2gS0$$
$$= u^2 - 2gH$$
$$H = \frac{u^2}{2g}$$

(vi) Total time to go up and return to the point of projection

Displacement S = 0 Thus from formula

$$S = ut - (1/2) gt^2$$
$$0 = ut - (1/2) gt^2$$
$$T = 2u/g$$

(vii) At any point C between A and B, where AC = s, the velocity v is given by

$$v = \pm\sqrt{u^2 - 2gS}$$

The velocity of body while crossing C upwards =

$$v = +\sqrt{u^2 - 2gS}$$

The velocity of body while crossing C downwards

$$v = -\sqrt{u^2 - 2gS}$$

Magnitudes of velocities are same

Solved numerical

Q) A body is projected upwards with a velocity 98 m/s.
Find (a) the maximum height reached
(b) the time taken to reach maximum height
(c) its velocity at height 196 m from the point of projection
(d) velocity with which it will cross down the point of projection and
(e) the time taken to reach back the point of projection.

Solution:

(a) Maximum height

$$H = \frac{u^2}{2g} = \frac{(98)^2}{2 \times 9.8} = 490\ m$$

(b) Time taken to reach maximum height
T= u/g = 9.8/9.8 = 10s

(c) Velocity at a height of 196 m from the point of projection

$$v = \pm\sqrt{u^2 - 2gS}$$

$$v = \pm\sqrt{(98)^2 - 2(9.8)(196)} = \pm 75.91\ m/s$$

+75.91 m/s while crossing the height upward and -75.91 m/ while crossing it downwards

(d)Velocity with which it will cross down the point of projection
Magnitude is same but direction is opposite hence V = -u =-98 m/s

(e)The time taken to reach back the point of projection

T= 2u/g = (2×98)/9.8 = 20 s

NUCLEUS

Nucleus

The nucleus consists of the elementary particles, protons and neutrons which are known as nucleons. A proton has positive charge of the same magnitude as that of electron and its rest mass is about 1836 times the mass of an electron. A neutron is electrically neutral, whose mass is almost equal to the mass of the proton. The nucleons inside the nucleus are held together by strong attractive forces called nuclear forces.

A nucleus of an element is represented as $_ZX^A$,

Where, X = Chemical symbol of the element.

Z = Atomic number which is equal to the number of protons

A = Mass number which is equal to the total number of protons and neutrons.

The number of neutrons is represented as N which is equal to A−Z.

For example: The chlorine nucleus is represented as $_{17}Cl^{35}$. It contains 17 protons and 18 neutrons.

Atomic mass is expressed in atomic mass unit (u), defined as (1/12)th of the mass of the carbon (C^{12}) atom. According to this definition.

$$1u = \frac{1.992647 \times 10^{-26}}{12} kg$$

$$1u = 1.660539 \times 10^{-27} \text{ kg}$$

Classification of nuclei

(i) Isotopes

Isotopes are atoms of the same element having the same atomic number Z but different mass number A. The nuclei $_1H^1$, $_1H^2$ and $_1H^3$ are the isotopes of hydrogen. As the atoms of isotopes have identical electronic structure, they have identical chemical properties and placed in the same location in the periodic table.

The relative abundance of different isotopes differs from element to element. Chlorine, for example, has two isotopes having masses 34.98 u and 36.98 u, which are nearly integral multiples of the mass of a hydrogen atom. The relative abundances of these isotopes are 75.4 and 24.6 per cent, respectively. Thus, the average mass of a chlorine atom is obtained by the weighted average of the masses of the two isotopes, which works out to be

$$= \frac{(75.4 \times 34.98) + (24.6 \times 36.98)}{100} = 35.47\ u$$

(ii) Isobars

Isobars are atoms of different elements having the same mass number A, but different atomic number Z. The nuclei $_8O^{16}$ and $_7N^{16}$ represent two isobars. Since isobars are atoms of different elements, they have different physical and chemical properties.

(iii) Isotones

Isotones are atoms of different elements having the same number of neutrons. $_6C^{14}$ and $_8O^{16}$ are some examples of isotones.

(iv) Isomers

For some nuclei Z values are same and A values are also same but their radioactive properties are different. They are called isomers of each other. $_{35}B^{80}$ has one pair of isomers

Discovery of Neutron

James Chadwick who observed emission of neutral radiation when beryllium nuclei were bombarded with alpha-particles. (α-particles are helium nuclei).
It was found that this neutral radiation could knock out protons from light nuclei such as those of helium, carbon and nitrogen.
Application of the principles of conservation of energy and momentum showed that if the neutral radiation consisted of photons, the energy of photons would have to be much higher than is available from the bombardment of beryllium nuclei with α-particles.
The clue to this puzzle, which Chadwick satisfactorily solved, was to assume that the neutral radiation consists of a new type of neutral particles called *neutrons*.
From conservation of energy and momentum, he was able to determine the mass of new particle 'as very nearly the same as mass of proton'. Mass of neutron m_n = 1.00866 u
Or 1.6749×10^{-27} kg

General properties of nucleus

Nuclear size
According to Rutherford's α−particle scattering experiment, the distance of the closest approach of α − particle to the nucleus was taken as a measure of nuclear radius, which is approximately 10^{-15}m.
If the nucleus is assumed to be spherical, an empirical relation is found to hold good between the radius of the nucleus R and its mass number A. It is given by

$$R = R_0 A^{\frac{1}{3}}$$

Where, $R_0 = 1.2 \times 10^{-15}$ m. or is equal to 1.2 F (1 Fermi, F = 10^{-15} m)

This means the volume of the nucleus, which is proportional to R^3 is proportional to A. Thus the density of nucleus is a constant, independent of A.

Nuclear density

The nuclear density ρ_N can be calculated from the mass and size of the nucleus

$$\rho_N = \frac{nuclear\ mass}{nuclear\ volume}$$

Nuclear mass = Am_N

where, A = mass number

m_N = mass of one nucleon and is approximately equal to 1.67×10^{-27} kg

Nuclear volume V_N

$$V_N = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi \left(R_0 A^{\frac{1}{3}}\right)^3$$

$$\rho_N = \frac{Am_N}{\frac{4}{3}\pi \left(R_0 A^{\frac{1}{3}}\right)^3} = \frac{m_N}{\frac{4}{3}\pi R_0^3}$$

Substituting the known values, the nuclear density is calculated as 1.816×10^{17} kg m^{-3} which is almost a constant for all the nuclei irrespective of its size. The high value of the nuclear density shows that the nuclear matter is in an extremely compressed state.

Nuclear mass

As the nucleus contains protons and neutrons, the mass of the nucleus is assumed to be the mass of its constituents.

Assumed nuclear mass = $Zm_P + Nm_N$,

Where, m_p and m_N are the mass of a proton and a neutron respectively

Z = number of protons

N = number of neutrons

However, from the measurement of mass by mass spectrometers, it is found that the mass of a stable nucleus (m) is less than the total mass of the nucleons.

i.e mass of a nucleus, $m < (Zm_p + Nm_N)$

$Zm_p + Nm_N - m = \Delta m$

where Δm is the mass defect

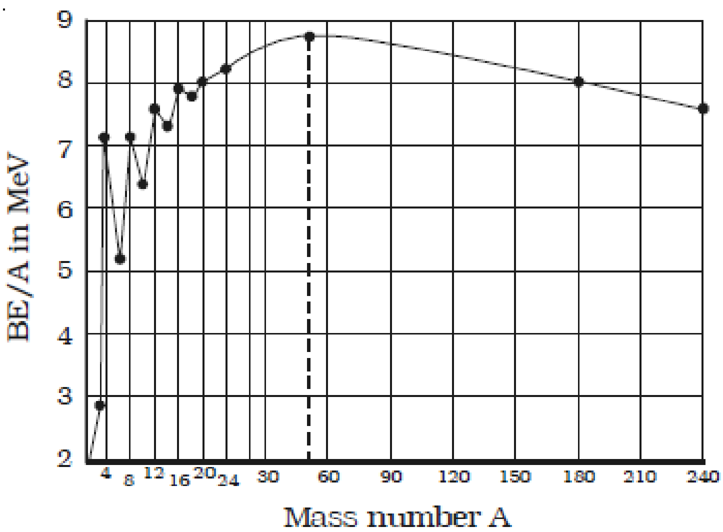
Thus, the difference in the total mass of the nucleons and the actual mass of the nucleus is known as the mass defect.

Note : In any mass spectrometer, it is possible to determine only the mass of the atom, which includes the mass of Z electrons.
If M represents the mass of the atom, then the mass defect can be written as

$$\Delta m = Zm_P + Nm_N + Zm_e - M$$

energy equivalent of 1 amu = 931 MeV

Binding energy



When the protons and neutrons combine to form a nucleus, the mass that disappears (mass defect, Δm) is converted into an equivalent amount of energy (Δmc^2). This energy is called the binding energy of the nucleus.
∴ Binding energy = $[Zm_P + Nm_N - m]\, c^2$
Binding energy = $\Delta m\, c^2$
The binding energy of a nucleus determines its stability against disintegration. In other words, if the binding energy is large, the nucleus is stable and vice versa.
The binding energy per nucleon is

$$\frac{BA}{A} = \frac{Binding\ energy\ of\ nucleus}{Total\ number\ of\ nucleons}$$

It is found that the binding energy per nucleon varies from element to element. A graph is plotted with the mass number A of the nucleus along the X−axis and binding energy per nucleon along the Y-axis.

Explanation of binding energy curve

(i) The binding energy per nucleon increases sharply with mass number A upto 20. It increases slowly after A = 20.
For A<20, there exists recurrence of peaks corresponding to those nuclei, whose mass numbers are multiples of four and they contain not only equal but also even number of protons and neutrons. Example: $_2He^4$, $_4Be^8$, $_6C^{12}$, $_8O^{16}$, and $_{10}Ne^{20}$.
The curve becomes almost flat for mass number between 30 and 170. Beyond 170, it decreases slowly as A increases.

(ii) The binding energy per nucleon reaches a maximum of 8.8 MeV at A=56, corresponding to the iron nucleus ($_{26}Fe^{56}$). Hence, iron nucleus is the most stable.

(iii) The average binding energy per nucleon is about 8.5 MeV for nuclei having mass number ranging between 30 and 170. These elements are comparatively more stable and non radioactive.

(iv) For higher mass numbers the curve drops slowly and the BE/A is about 7.6 MeV for uranium. Hence, they are unstable and radioactive.

(v) The lesser amount of binding energy for lighter and heavier nuclei explains nuclear fusion and fission respectively. A large amount of energy will be liberated if lighter nuclei are fused to form heavier one (fusion) or if heavier nuclei are split into lighter ones (fission).

<u>Nuclear force</u>

The nucleus of an atom consists of positively charged protons and uncharged neutrons. According to Coulomb's law, protons must repel each other with a very large force, because they are close to each other and hence the nucleus must be broken into pieces. But this does not happen. It means that, there is some other force in the nucleus which overcomes the electrostatic repulsion between positively charged protons and binds the protons and neutrons inside the nucleus. This force is called nuclear force.

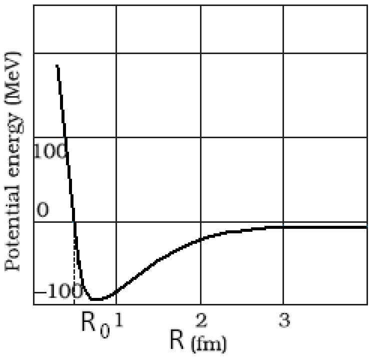
(i) Nuclear force is charge independent. It is the same for all the three types of pairs of nucleons (n−n), (p−p) and (n−p). This shows that nuclear force is not electrostatic in nature

(ii) Nuclear force is the strongest known force in nature. Nuclear force is about 1040 times stronger than the gravitational force.

(iii) Nuclear force is a short range force. It is very strong between two nucleons which are less than 10^{-15} m apart and is almost negligible at a distance greater than this. On the other hand electrostatic, magnetic and gravitational forces are long range forces that can be felt easily.

The nuclear force between two nucleons falls rapidly to zero as their distance is more than a few femtometres. This leads to *saturation of forces* in a medium or a large-sized nucleus, which is the reason for the constancy of the binding energy per nucleon.

A rough plot of the potential energy between two nucleons as a function of distance is shown in the Fig.



The potential energy is a minimum at a distance R_0 of about 0.8 fm. This means that the force is attractive for distances larger than 0.8 fm and repulsive if they are separated by distances less than 0.8 fm.

However, the present view is that the nuclear force that binds the protons and neutrons is not a fundamental force of nature but it is secondary.

Radioactivity

The phenomenon of spontaneous emission of highly penetrating radiations such as α, β and γ rays by heavy elements having atomic number greater than 82 is called radioactivity and the substances which emit these radiations are called radioactive elements. The radioactive phenomenon is spontaneous and is unaffected by any external agent like temperature, pressure, electric and magnetic fields etc.

Experiments performed showed that radioactivity was a nuclear phenomenon in which an unstable nucleus undergoes a decay. This is referred to as *radioactive decay*. Three types of radioactive decay occur in nature :

(i) α-decay in which a helium nucleus $_2He^4$ is emitted.

(ii) β-decay in which electrons or positrons (particles with the same mass as electrons, but with a charge exactly opposite to that of electron) are emitted;

(iii) γ-decay in which high energy (hundreds of keV or more) photons are emitted.

Each of these decay will be considered in subsequent sub-sections

Law of radioactive decay

In any radioactive sample, which undergoes α, β or γ-decay, it is found that the number of nuclei undergoing the decay per unit time is proportional to the total number of nuclei in the sample. If *N* is the number of nuclei in the sample and Δ*N* undergo decay in time Δ*t* then

$$\frac{\Delta N}{\Delta t} \propto N$$

$$\frac{\Delta N}{\Delta t} = -\lambda N$$

where λ is called the radioactive *decay constant* or *disintegration constant*.
The change in the number of nuclei in the sample is dN = – ΔN in time Δt. Thus the rate of change of N is (in the limit Δt → 0)

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\lambda N - - - (1)$$

$$\frac{\mathrm{d}N}{N} = -\lambda \mathrm{d}t$$

Now, integrating both sides of the above equation, we get,

$$\int_{N_0}^{N} \frac{\mathrm{d}N}{N} = -\lambda \int_{t_0}^{t} dt$$

$$\ln N - \ln N_0 = -\lambda (t - t_0)$$

Here N_0 is the number of radioactive nuclei in the sample at some arbitrary time t0 and N is the number of radioactive nuclei at any subsequent time t. Setting t_0 = 0 and rearranging Equation gives us

$$ln \frac{N}{N_0} = -\lambda t$$

$$N = N_0 e^{-\lambda t} \quad - - - (2)$$

Above equation represents law of radioactive decay
Differentiating equation (2) we get

$$\frac{dN}{dt} = -\lambda N_0 e^{-\lambda t}$$

$$-\frac{dN}{dt} = \lambda N_0 e^{-\lambda t}$$

Term -dN/dt is called the rate of disintegration or activity I of element at time t
From equation (1) we get I = λN
Thus

$$I = I_0 e^{-\lambda t}$$

Is alternative form of the law of *law of radioactive decay*
The SI unit for activity is becquerel, named after the discoverer of radioactivity, Henry Becquerel. It is defined as

1 becquerel = 1Bq = 1 decay per second

An older unit, the curie, is still in common use:

1 curie = 1 Ci = 3.7×10^{10} Bq (decays per second

Half life period

The half life period of a radioactive element is defined as the time taken for one half of the radioactive element to undergo disintegration.

From the law of disintegration

$$N = N_0 e^{-\lambda t}$$

Let $T_{½}$ be the half life period. Then, at $t = T_{½}$, $N = N_0/2$

$$\frac{N_0}{2} = N_0 e^{\lambda T_{1/2}}$$

$$\ln 2 = \lambda T_{½}$$

$$T_{1/2} = \frac{ln2}{\lambda}$$

$$T_{1/2} = \frac{0.693}{\lambda}$$

Fraction of radioactive substance left undecayed is,

$$\frac{N}{N_0} = (\frac{1}{2})^n$$

where n is the number of half lives.

$$n = \frac{total\ time}{half\ life}$$

The half life period is inversely proportional to its decay constant. For a radioactive substance, at the end of T½, 50% of the material remain unchanged. After another T½ i.e., at the end of 2 T½, 25% remain unchanged. At the end of 3 T½, 12.5% remain unchanged and so on.

Solved Numerical

1. The half life of radon is 3.8 days. After how many days 19/20 of the sample will decay

Solution

If we take 20 parts as N_0 then N=1

From formula

$$\frac{N}{N_0} = (\frac{1}{2})^n$$

$$\frac{1}{20} = (\frac{1}{2})^n$$

$$20 = 2^n$$

using log we get log20=nlog2

1.3010 = n× 0.3010 thus n = 4.322

From formula

$$n = \frac{total\ time}{half\ life}$$

$$4.322 = \frac{t}{3.8}$$

t=16.42 days

Q) An archaeologist analysis of the wood in a prehistoric structure revels that the ratio of ^{14}C (half life = 5700 years) to ordinary carbon is only one fourth in the cells of living plants. What is the age of the wood?

Solution:

If we take $N_0 = 1$ then N = ¼

From formula

$$N = N_0 e^{-\lambda t}$$

$$\frac{N}{N_0} = e^{-\lambda t}$$

$$\frac{1}{4} = e^{-\lambda t}$$

$$4 = e^{\lambda t}$$

Taking log to the base e on both sides

ln4 = λt

Converting to log to base 10

2.303 log4 = λt

From formula for half life

$$\lambda = \frac{0.693}{T_{1/2}}$$

$$2.303\ log4 = \frac{0.693}{T_{1/2}} t$$

$$t = \frac{2.303\ log4 \times T_{1/2}}{0.693}$$

$$t = \frac{2.303 \times 0.6021 \times 5700}{0.693}$$

t =11400 years

Q) A radioactive nucleus X decays to nucleus Y with a decay constant $\lambda_X = 0.1\ s^{-1}$.

Y further decays to a stable nucleus Z with decay constant $\lambda_Y = \frac{1}{30} s^{-1}$

Initially there are only X nuclei and their number is $N_0 = 10^{20.}$

Set up the rate equation for the population of X, Y, and Z. The population of the Y nucleus as function of time is given by

$$N_Y = \frac{N_0\lambda_X}{\lambda_X - \lambda_Y}(e^{-\lambda_y t} - e^{-\lambda_x t})$$

Find the time at which N_Y is maximum and determine the population X and Z at that instant

Solution

Rate equation for X from Law of radioactive decay

$$\frac{dN_X}{dt} = -\lambda_X N_X \quad --- eq(1)$$

Rate of decay of Y depends on generation of Y due to decay of X and population of Y at that instant thus

$$\frac{dN_Y}{dt} = \lambda_X N_X - \lambda_y N_y \quad --- eq(2)$$

Rate of disintegration of Z depends only on rate of generation of Y thus

$$\frac{dN_Z}{dt} = \lambda_Y N_Y \quad --- eq(3)$$

For N_Y to be maximum eq(2) should become zero

$$\lambda_X N_X - \lambda_y N_y = 0$$

$$\lambda_X N_X = \lambda_y N_y --- eq(4)$$

We know that

$$N_X = N_0 e^{-\lambda_x t} --- eq(5)$$

Given

$$N_Y = \frac{N_0\lambda_X}{\lambda_X - \lambda_Y}(e^{-\lambda_y t} - e^{-\lambda_x t}) --- eq(6)$$

Substituting values of N_X and N_Y from equation (5) and (6) in equation (4) we get

$$\lambda_X N_0 e^{-\lambda_x t} = \lambda_y \frac{N_0\lambda_X}{\lambda_X - \lambda_Y}(e^{-\lambda_y t} - e^{-\lambda_x t})$$

$$e^{-\lambda_x t} = \lambda_y \frac{1}{\lambda_X - \lambda_Y}(e^{-\lambda_y t} - e^{-\lambda_x t})$$

$$\frac{\lambda_X - \lambda_Y}{\lambda_y} = \frac{(e^{-\lambda_y t} - e^{-\lambda_x t})}{e^{-\lambda_x t}}$$

$$\frac{\lambda_X}{\lambda_y} - 1 = \frac{e^{-\lambda_y t}}{e^{-\lambda_x t}} - 1$$

$$\frac{\lambda_X}{\lambda_y} = e^{(\lambda_x t - \lambda_y t)}$$

Taking log on both side

$$(\lambda_x - \lambda_y)t = ln\left(\frac{\lambda_X}{\lambda_y}\right)$$

$$t = \frac{1}{(\lambda_x - \lambda_y)} ln\left(\frac{\lambda_X}{\lambda_y}\right)$$

$$t = \frac{1}{0.1 - \frac{1}{30}} ln\left(\frac{0.1}{1/30}\right)$$

$$t = 15ln(3)$$
$$t = 2.303 \times 15 \times log(3)$$

t=16.48s is time when population of Y is maximum

To find population of X and Z at t = 16.48s

We will use equation

$$N_X = N_0 e^{-\lambda_x t}$$

$$N_X = 10^{20} \times e^{-0.1 \times 16.48} = 10^{20} \frac{1}{e^{1.648}}$$

[Calculation of $e^{1.648}$

$\log_{10}(e^{1.648})$ =(1.648)$\log_{10}e$

$\log_{10}(e^{-1.68})$ =(1.648) × 0.434 = 0.7155

antilog (0.7155) = 5.194

thus value of $e^{1.648}$ = 5.194]

$\therefore N_X = 10^{20} \times \frac{1}{5.194} = 1.925 \times 10^{19}$

From equation (4)

$$\lambda_X N_X = \lambda_y N_y$$
$$N_Y = N_X \frac{\lambda_X}{\lambda_y}$$

$$N_Y = 1.925 \times 10^{19} \times \frac{0.1}{1/30} = 3 \times 1.925 \times 10^{19} = 5.772 \times 10^{19}$$

Now $N_Z = N_0 - N_X - N_Y$

$N_z = (10\times10^{19}) - (1.925\times10^{19}) - (5.772\times10^{19}) = 2.303\times10^{19}$

Q) In a mixture of two elements A and B having decay constants 0.1 day^{-1} and 0.2 day^{-1} respectively; initially the activity of A is 3 times that of B. If the initial activity of the mixture is 2mCi, find the activity of it after 10 days

Solution:

λ_A= 0.1 day^{-1} λ_B=0.2 day^{-1}

$(I_0)_A = 3(I_0)_B$

At time t = 0, activity of mixture is

$I_0 = (I_0)_A + (I_0)_B$

$I_0 = 3(I_0)_B + (I_0)_B$

$2 = 4(I_0)_B$
$(I_0)_B = 0.5$ mCi
$(I_0)_A = 1.5$ mCi
At time t , activity of A is

$$I_A = (I_0)_A e^{-\lambda_A t} = (1.5)e^{-(0.1)(10)}$$

$$I_A = \frac{1.5}{e} = \frac{1.5}{2.718} = 0.552\ mCi$$

At time t , activity of B is

$$I_B = (I_0)_B e^{-\lambda_B t} = (0.5)e^{-(0.2)(10)}$$

$$I_B = \frac{1.5}{e^2} = \frac{1.5}{(2.718)^2} = 0.067\ mCi$$

At time t, total activity of the mixture
$I = I_A + I_B = 0.552 + 0.067 = 0.619$ mCi

Mean life (τ)

The time-interval, during which the number of nuclei of a radioactive element becomes equal to the e^{th} part of its original number, is called the mean life or average life τ of the element .
When $N = N_0/e$, we can put t = mean life = τ

$$\therefore \frac{N_0}{e} = N_0\, e^{-\lambda\tau}$$

$$e = e^{\lambda\tau}$$

$$\tau = \frac{1}{\lambda}$$

Thus mean life is equal to the reciprocal of the decay constant
Relation between $T_{1/2}$ and mean life

$$T_{1/2} = \frac{0.693}{\lambda}$$

$$T_{1/2} = 0.693\tau$$

Solved Numerical

Q) A radioactive sample emits n β particles in 2 second. In next 2 seconds, it emits 0.75n β particles. What is the mean life of the sample

Solution:

Disintegration of one nucleon give one β particle
If n β-particles are emitted then N-n nucleons are not disintegrated thus
$N - n = Ne^{-\lambda 2}$
$n = N(1 - e^{-2\lambda})$-----eq(1)

In 4 seconds total emission o is n +0.75n = (1.75)n thus

$(1.75)n = N(1-e^{-4\lambda})$ ---eq(2)

Dividing eq(2) by eq(1)

$$1.75 = \frac{1-e^{-4\lambda}}{1-e^{-2\lambda}}$$
$$\frac{7}{4} = \frac{1-e^{-4\lambda}}{1-e^{-2\lambda}}$$
$$7-7e^{-2\lambda} = 4-4e^{-4\lambda}$$
$$4e^{-4\lambda}-7e^{-2\lambda}+3=0$$
$$4e^{-4\lambda}-4e^{-2\lambda}-3e^{-2\lambda}+3=0$$
$$4e^{-2\lambda}(e^{-2\lambda}-1)-3(e^{-2\lambda}-1)=0$$
$$(e^{-2\lambda}-1)(4e^{-2\lambda}-3)=0$$
$$\text{But } e^{-2\lambda}-1 \neq 0$$
$$\therefore 4e^{-2\lambda}-3=0$$
$$e^{-2\lambda}=\frac{3}{4}$$
$$e^{2\lambda}=\frac{4}{3}$$
$$2\lambda = ln\left(\frac{4}{3}\right)$$
$$\frac{1}{\lambda}=\frac{2}{ln\left(\frac{4}{3}\right)}=\frac{2}{ln4-ln3}$$
$$\tau=\frac{2}{ln4-ln3}$$

Radioactive displacement law

During a radioactive disintegration, the nucleus which undergoes disintegration is called a parent nucleus and that which remains after the disintegration is called a daughter nucleus. In 1913,Soddy and Fajan framed the displacement laws governing radioactivity.

α-decay

When a radioactive nucleus disintegrates by emitting an α-particle, the atomic number decreases by two and mass number decreases by four. The α-decay can be expressed as

${}_{z}X^{A} \rightarrow {}_{z-2}Y^{A-4} + {}_{2}He^{4}$

example, when ${}_{92}U^{238}$ undergoes alpha-decay, it transforms to ${}_{90}Th^{234}$

$$\mathbf{{}_{92}U^{238} \rightarrow {}_{90}Th^{234} + {}_{2}He^{4}}$$

The alpha-decay of ${}_{92}U^{238}$ can occur spontaneously (without an external source of energy) because the total mass of the decay products ${}_{90}Th^{234}$ and 2He4 is less than the mass of the original ${}_{92}U^{238}$.

Thus, the total mass energy of the decay products is less than the mass energy of the original nuclide.
The difference between the initial mass energy and the final mass energy of the decay products is called the *Q value* of the process or the *disintegration energy*.
Thus, the *Q* value of an alpha decay can be expressed as

$$Q = (m_X - m_Y - m_{He})c^2$$

This energy is shared by the daughter nucleus and the alpha-particle, in the form of kinetic energy. Alpha-decay obeys the radioactive law

β–*decay*

In the process of β-decay, a nucleus spontaneously emits electron or positron. Positron has the same charge as that of electron but it is positive and its other properties are exactly identical to those of electron. Thus positron and the antiparticle of electron. Positron and electron are respectively written as β^+ and β^- or ${}_{+1}e^0$ and ${}_{-1}e^0$ and e^+ and e^-

β^- emission

$${}_{15}P^{32} \rightarrow {}_{16}S^{32} + {}_{-1}e^0 + \bar{\upsilon}$$ ($\bar{\upsilon}$ is called antineutrino)

Compared to parent element, the atomic number of daughter element is one unit more in β^- decay
In this reaction neutron disintegrates into proton can be sated as

$$N^0 \rightarrow P^{+1} + \beta^- + \bar{\upsilon}$$

β^+ emission

$${}_{11}Na^{22} \rightarrow {}_{10}Ne^{23} + {}_{+1}e^0 + \upsilon$$ (υ is called neutrino)

Compared to parent element, the atomic number of daughter element is one unit less in β^+ decay
In this reaction Proton disintegrates into Neutron can be sated as

$$P^{+1} \rightarrow N^0 + \beta^+ + \upsilon$$

Neutrino and anti-neutrino are the anti particles of each other. They are electrically neutral and their mass is extremely small as compared to even that of electron. Their interaction with other particles is negligible and hence it is extremely difficult to detect them. They can pass without interaction even through very large matter (even through the entire earth). They have h/2π spin

γ–decay

There are energy levels in a nucleus, just like there are energy levels in atoms. When a nucleus is in an excited state, it can make a transition to a lower energy state by the emission of electromagnetic radiation.
As the energy differences between levels in a nucleus are of the order of MeV, the photons emitted by the nuclei have MeV energies and are called **gamma rays**.
Most radio-nuclides after an alpha decay or a beta decay leave the daughter nucleus in an excited state. The daughter nucleus reaches the ground state by a single transition or sometimes by successive transitions by emitting one or more gamma rays.
A well-known example of such a process is that of $_{27}Co^{60}$.
By beta emission, the $_{27}Co^{60}$ nucleus transforms into $_{28}Ni^{60}$ nucleus in its excited state.
The excited $_{28}Ni^{60}$ nucleus so formed then de-excites to its ground state by successive emission of 1.17 MeV and 1.33 MeV gamma rays.

Nuclear reactions

By bombarding suitable particles of suitable energy on a stable element, that element can be transformed into another element. Such a reaction is called artificial nuclear transmutation. Example

$$_{7}N^{14} + {_{2}He^{4}} \rightarrow {_{8}O^{17}} + {_{1}H^{1}} + Q$$

Such process, in which change in the nucleus takes place are called nuclear reactions. Here Q is called Q –value of the nuclear reaction and it shows that the energy released in the process. If Q>0, the reaction is exoergic and if Q<0 then reaction is endoergic
Reaction can be symbolically represented as

$$A + a \rightarrow B + b + Q$$

A : is called the target nucleus
a: is called projectile partile
B: is called product nucleus
b: is called emitted particle

The energy liberated $Q = [m_A + m_a - m_B - m_b]\, c^2$, here m represents the mass of respective particle

Nuclear Fission

The process of breaking up of the nucleus of a heavier atom into two fragments with the release of large amount of energy is called nuclear fission.

The fission is accompanied of the release of neutrons.
The fission reactions with $_{92}U^{235}$ are represented as

$_{92}U^{235} + {_0}n^1 \rightarrow {_{56}}Ba^{141} + {_{36}}Kr^{92} + 3\ {_0}n^1 + Q$ ...(1)
$_{92}U^{235} + {_0}n^1 \rightarrow {_{54}}Xe^{140} + {_{38}}Sr^{94} + 2\ {_0}n^1 + Q$ ...(2)
$_{92}U^{235} + {_0}n^1 \rightarrow {_{51}}Sb^{140} + {_{41}}Nb^{99} + 4\ {_0}n^1 + Q$ ...(3)

The product nuclei obtained by the fission are called the fission fragments, the neutrons are called the fission neutrons and energy is called fission energy. In the above reaction 60 different nuclei are obtained as fission fragment, having Z value between 36 and 56. The probability is maximum for formation of nucli with A = 95 and A = 140. The fission fragments are radio active and by successive emission of β^- particles results in stable nucli. The disintegration energy in fission events first appears as the kinetic energy of the fragments and neutrons. Eventually it is transferred to the surrounding matter appearing as heat.

Solved Numerical

Q In the reaction $_ZX^A \rightarrow {_{z-2}}Y^{A-4} + {_2}He^4 + Q$ of the nucleus X at rest, taking the ratio of mass of α-particle M_α and mass of Y-nucleus as

$$\frac{M_\alpha}{M_\beta} = \frac{4}{A-4}$$

Show that the Q-value of the reaction is given by

$$Q = K_\alpha \left(\frac{A}{A-4}\right)$$

K_α = kinetic energy of α particles

Solution:

Q-value of reaction = energy equivalent to mass-difference

$$Q = (M_X - M_Y - M_\alpha)c^2$$

Q = increase in kinetic energy

Q = ($K_\alpha + K_\beta$) – 0 (∴ X was steady)

$$Q = \frac{1}{2}M_\alpha v_\alpha^2 + \frac{1}{2}M_\beta v_\beta^2 \quad ----eq(1)$$

From conservation of momentum

$$M_Y\vec{v}_Y + M_Y\vec{v}_Y = 0$$

$M_Y v_Y = M_\alpha v_\alpha$ (in magnitude)

$$v_Y = \left(\frac{M_\alpha}{M_Y}\right) v_\alpha$$

Substituting this value in equation (1)

$$Q = \frac{1}{2}M_\alpha v_\alpha^2 + \frac{1}{2}M_Y \left(\frac{M_\alpha}{M_Y}\right)^2 v_\alpha^2$$

$$Q = \frac{1}{2}M_\alpha v_\alpha^2 \left[\frac{M_\alpha}{M_Y} + 1\right]$$

$$Q = K_\alpha \left[\frac{4}{A-4} + 1\right]$$

$$Q = K_\alpha \left(\frac{A}{A-4}\right)$$

Chain reaction

Consider a neutron causing fission in a uranium nucleus producing three neutrons. The three neutrons in turn may cause fission in three uranium nuclei producing nine neutrons. These nine neutrons in turn may produce twenty seven neutrons and so on. A chain reaction is a self propagating process in which the number of neutrons goes on multiplying rapidly almost in a geometrical progression.

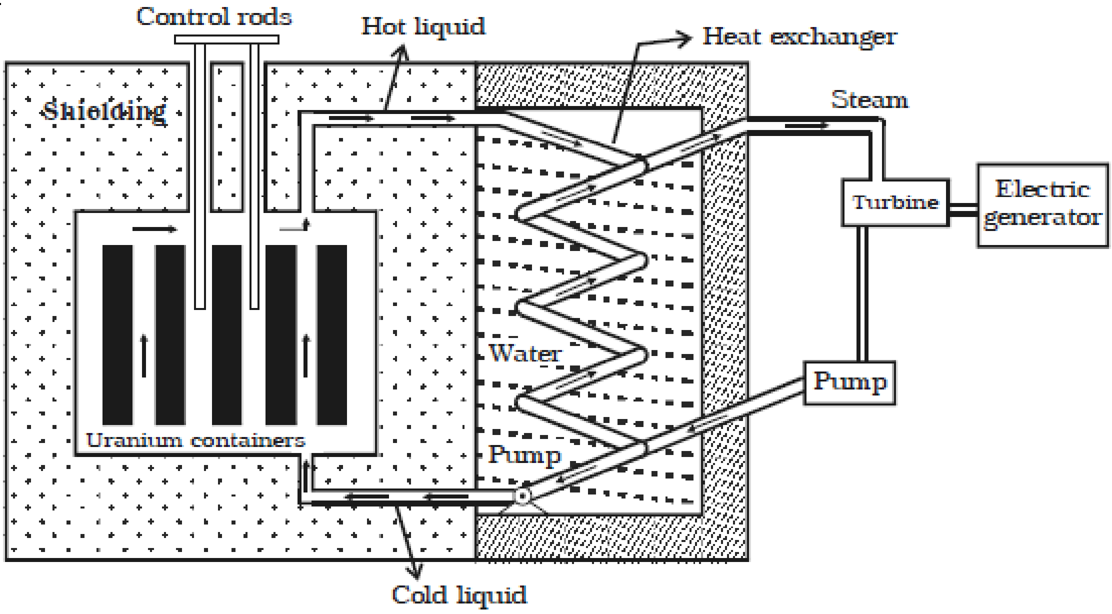
Critical size

Critical size of a system containing a fissile material is defined as the minimum size in which at least one neutron is available for further fission reaction. The mass of the fissile material at the critical size is called critical mass. The chain reaction is not possible if the size is less than the critical size.

Nuclear reactor

A nuclear reactor is a device in which the nuclear fission reaction takes place in a self sustained and controlled manner

The schematic diagram of a nuclear reactor is shown in Fig In such a reactor, water is used both as the moderator and as the heat transfer medium. In the *primary-loop*, water is circulated through the reactor vessel and transfers energy at high temperature and pressure (at about 600 K and 150 atm) to the steam generator, which is part of the *secondary-loop*. In the steam generator, evaporation provides high-pressure steam to operate the turbine that drives the electric generator. The low-pressure steam from the turbine is cooled and condensed to water and forced back into the steam generator



(i) Fissile material or fuel

The fissile material or nuclear fuel generally used is $_{92}U^{235}$. But this exists only in a small amount (0.7%) in natural uranium. Natural uranium is enriched with more number of $_{92}U^{235}$ (2 – 4%) and this low enriched uranium is used as fuel in some reactors. Other than U^{235}, the fissile isotopes U^{233} and Pu^{239} are also used as fuel in some of the reactors.

(ii) Moderator

The function of a moderator is to slow down fast neutrons produced in the fission process having an average energy of about 2 *MeV* to thermal neutrons with an average energy of about 0.025 *eV*, which are in thermal equilibrium with the moderator. Ordinary water and heavy water(D_2O) are the commonly used moderators. A good moderator slows down neutrons by elastic collisions and it does not remove them by absorption. The moderator is present in the space between the fuel rods in a channel. Graphite is also used as a moderator in some countries. In fast breeder reactors, the fission chain reaction is sustained by fast neutrons and hence no moderator is required.

(iii) Neutron source

A source of neutron is required to initiate the fission chain reaction for the first time. A mixture of beryllium with plutonium or radium or polonium is commonly used as a source of neutron.

(iv) Control rods

The control rods are used to control the chain reaction. They are very good absorbers of neutrons. The commonly used control rods are made up of elements like boron or cadmium. The control rods are inserted into the core and they pass through the space in between the fuel tubes and through the moderator. By pushing them in or pulling out, the reaction rate can be controlled. In our country, all the power reactors use boron carbide (B4C), a ceramic material as control rod.

Because of the use of control rods, it is possible that the ratio, *K*, of number of fission produced by a given generation of neutrons to the number of fission of the preceeding generation may be greater than one. This ratio is called the *multiplication factor;* it is the measure of the growth rate of the neutrons in the reactor. For $K = 1$, the operation of the reactor is said to be *critical*, which is what we wish it to be for steady power operation. If *K* becomes greater than one, the reaction rate and the reactor power increases exponentially. Unless the factor *K* is brought down very close to unity, the reactor will become supercritical and can even explode.

In addition to control rods, reactors are provided with *safety rods* which, when required, can be inserted into the reactor and *K* can be reduced rapidly to less than unity.

(v) The cooling system

The cooling system removes the heat generated in the reactor core. Ordinary water, heavy water and liquid sodium are the commonly used coolants. A good coolant must possess large specific heat capacity and high boiling point. The coolant passes through the tubes containing the fuel bundle and carries the heat from the fuel rods to the steam generator through heat exchanger. The steam runs the turbines to produce electricity in power reactors.
Being a metal substance, liquid sodium is a very good conductor of heat and it remains in the liquid state for a very high temperature as its boiling point is about 1000^o C.

(vi) Neutron reflectors
Neutron reflectors prevent the leakage of neutrons to a large extent, by reflecting them back. In pressurized heavy water reactors the moderator itself acts as the reflector.
In the fast breeder reactors, the reactor core is surrounded by depleted uranium (uranium which contains less than 0.7% of ${}_{92}U^{235}$) or thorium (${}_{90}Th^{232}$) which acts as neutron reflector. Neutrons escaping from the reactor core convert these materials into Pu^{239} or U^{233} respectively.

(vii) Shielding
As a protection against the harmful radiations, the reactor is surrounded by a concrete wall of thickness about 2 to 2.5 m.

Nuclear fusion – energy generation in stars

Energy can be released if two light nuclei combine to form a single larger nucleus, a process called *nuclear fusion*.
Some examples of such energy liberating reactions are

$${}_1H^1 + {}_1H^1 \rightarrow {}_1H^2 + e^+ + \nu + 0.42 \text{ MeV} \quad \text{-- (a)}$$
$$1H^2 + 1H^2 \rightarrow {}_2He^3 + n + 3.27 \text{ MeV} \quad \text{---(b)}$$
$$1H^2 + 1H^2 \rightarrow {}_1He^3 + {}_1H^1 + 4.03 \text{ MeV} \quad \text{--- (c)}$$

In all these reactions, we find that two positively charged particles combine to form a larger nucleus.
It must be realized that such a process is hindered by the Coulomb repulsion that acts to prevent the two positively charged particles from getting close enough to be within the range of their attractive nuclear forces and thus 'fusing'.
The height of this *Coulomb barrier* depends on the charges and the radii of the two interacting nuclei. The temperature at which protons in a proton gas would have enough energy to overcome the coulomb's barrier is about 3×10^9 K.
To generate useful amount of energy, nuclear fusion must occur in bulk matter. What is needed is to raise the temperature of the material until the particles have enough energy – due to their thermal motions alone – to penetrate the coulomb barrier. This process is called *thermonuclear fusion*.

The fusion reaction in the sun is a multi-step process in which hydrogen is burned into helium, hydrogen being the ‘fuel’ and helium the ‘ashes’.
The *proton-proton* (p, p) *cycle* by which this occurs is represented by the following sets of reactions:

$$\mathbf{{}_1H^1 + {}_1H^1 \rightarrow {}_1H^2 + e^+ + \nu + 0.42\ MeV\ \text{---}\ (d)}$$
$$\mathbf{e^+ + e^- \rightarrow \gamma + 1.02\ MeV\ \text{---}\ (e)}$$
$$\mathbf{{}_1H^2 + {}_1H^1 \rightarrow {}_2He^3 + \gamma + 5.49\ MeV\ \text{---}\ (f)}$$
$$\mathbf{{}_2H^3 + {}_2H^3 \rightarrow {}_2He^4 + {}_1H^1 + {}_1H^1 + 12.86\ MeV\ \text{---}(g)}$$

For the fourth reaction to occur, the first three reactions must occur twice, in which case two light helium nuclei unite to form ordinary helium or nucleus.
the net effect is

$$\mathbf{4\,{}_1H^1 + 2e^- \rightarrow {}_2He^4 + 2\nu + 6\gamma + 26.7MeV}$$

Thus, four hydrogen atoms combine to form an ${}_2He^4$ atom with a release of 26.7 MeV of energy.
Calculations show that there is enough hydrogen to keep the sun going for about the same time into the future. In about 5 billion years, however, the sun’s core, which by that time will be largely helium, will begin to cool and the sun will start to collapse under its own gravity. This will raise the core temperature and cause the outer envelope to expand, turning the sun into what is called a *red giant*
If the core temperature increases to 10^8 K again, energy can be produced through fusion once more – this time by burning helium to make carbon. As a star evolves further and becomes still hotter, other elements can be formed by other fusion reactions.
The energy generation in stars takes place via thermonuclear fusion.

Solved Numerical

Q) By the fusion of 1Kg deuterium ($1H^2$) according the reaction
${}_1H^2 + {}_1H^2 \rightarrow {}_2He^4 + {}_0n^1$ + 3.27 Mev, how long can a bulb of 100 W give light?
Molecular wt of deuterium is 2g
Thus number of moles of deuterium in 1kg = 500 moles
Number of nucli of deuterium = $500\times6.02\times10^{23} = 3.01\times10^{26}$
Now Two nucli gives energy of 3.27 MeV = $3.27\times10^6\times1.6\times10^{-19}$ J=5.23×10^{-13} J
Thus 3.01×10^{26} nucli will release energy of

$$\frac{3.01\times10^{26}\times5.23\times10^{-13}}{2} = 7.87\times10^{13}\,J$$

If a bulb of 100 W glows for t seconds, then energy consumed = 100t J

$$\therefore 100t = 7.87 \times 10^{13}$$

t = 7.874×10[11] sec

$$t = \frac{7.874 \times 10^{11}}{3.16 \times 10^{7\ s/year}} = 24917\ Yr$$

OSCILLATIONS

Periodic Motion and Oscillatory motion

If a body repeats its motion along a certain path, about a fixed point, at a definite interval of time, it is said to have a *periodic motion*

If a body moves to and fro, back and forth, or up and down about a fixed point in a definite interval of time, such motion is called an *oscillatory motion.* The body performing such motion is called an *oscillator*.

Simple Harmonic Motion

The periodic motion of a body about a fixed point, on a linear path, under the influence of the force acting towards the fixed point and proportional to displacement of the body from the fixed point is called a *simple harmonic motion* (SHM)

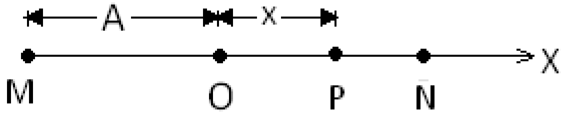
A body performing simple harmonic motion is known as *Simple Harmonic Oscillator (SHO)*

Simple harmonic motion is a special type of periodic motion in which

(i) The particle oscillates on a straight line.

(ii) The acceleration of the particle is always directed towards a fixed point on the straight line.

(iii) The magnitude of acceleration is proportional to the displacement of the particle from fixed point.



This fixed point is called the centre of the oscillation or *mean position*. Taking this point as origin "O".

The maximum displacement of oscillator on either side of the mean position is called *amplitude* denoted by A

The time required to complete one oscillation is known as *periodic time* (T) of oscillator

In other words, the least time interval of time after which the periodic motion of an oscillator repeat itself is called a periodic time of the oscillator. Distance travelled by oscillator is 4A in *periodic time*

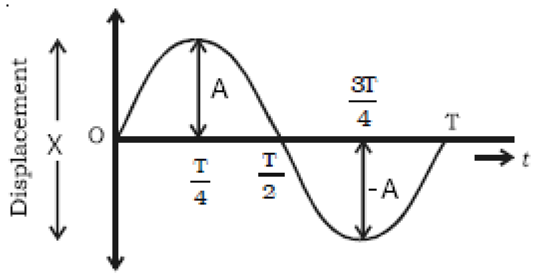
The number of oscillations completed by simple harmonic oscillator in one second is defined as *frequency*. SI unit is s^{-1} or hertz (H)

It is denoted by f and f = 1/T

2π times the frequency of an oscillator is called the *angular frequency* of the oscillator

It is denoted by ω. Its SI unit is rad s^{-1}, ω = 2πf.

If we draw the graph of displacement of SHO against time as shown in figure, which is as shown in figure



Mathematical equation is the function of time
x(t) = Asin(ωt +ϕ) -----eq(1)
Here X(t) represents displacement at time t
A = Amplitude
(ωt + ϕ) = Phase
ϕ = Initial phase (epoch) for a given graph ϕ= 0.
If oscillation have started from negative x end (M) then initial phase is –π/2
If oscillation stated from positive end (N) then initial phase is π/2
ω = angular frequency

Velocity

Velocity of oscillator is

$$v(t) = \frac{dx(t)}{dt}$$

v(t) = ωAcos(ωt + φ)

$$v(t) = \pm\omega A\ (\sqrt{1 - sin^2(\omega t + \varphi)})$$

$$v(t) = \pm\omega\ (\sqrt{A^2 - A^2 sin^2(\omega t + \varphi)})$$

From eq(1)

$$v(t) = \pm\ \omega\ (\sqrt{A^2 - x^2})$$

Velocity is maximum at x = 0 or equilibrium position v = ±ωA
Velocity is minimum at x = A or extreme positions v= 0
Note that velocity is out of phase of displacement by π/2

Acceleration

$$a(t) = \frac{dv(t)}{dt}$$

a(t) = $-\omega^2$Asin(ωt + ϕ)
a(t) = $-\omega^2$ x(t)
At x = 0 ,acceleration is a = 0

And maximum at x = ±A , a = ∓ $\omega^2 A$

Note velocity is out of phase of displacement by π

Solved Numerical

Q) A particle moving with S.H.M in straight line has a speed of 6m/s when 4m from the centre of oscillations and a speed of 8m/s when 3m from the centre. Find the amplitude of oscillation and the shortest time taken by the particle in moving from the extreme position to a point midway between the extreme position and the centre

Solution

From the formula for velocity

$$v(t) = \pm\,\omega\,(\sqrt{A^2 - x^2})$$

$$6 = \pm\,\omega\,(\sqrt{A^2 - 4^2})$$

And

$$8 = \pm\,\omega\,(\sqrt{A^2 - 3^2})$$

By taking ratio of above equations

$$\frac{6}{8} = \frac{\sqrt{A^2 - 4^2}}{\sqrt{A^2 - 3^2}}$$

On simplifying we get A = 5 m

On substituting value of A in equation

$$6 = \pm\,\omega\,(\sqrt{A^2 - 4^2})$$

$$6 = \pm\,\omega\,(\sqrt{5^2 - 4^2})$$

ω = 2 rad

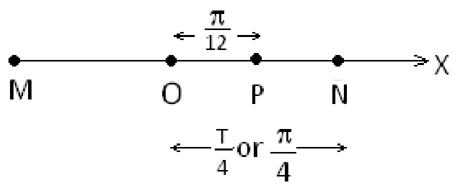
From the equation for oscillation

x(t) = Asin(ωt +ϕ)

$$\frac{A}{2} = A\sin 2t$$

$$0.5 = \sin 2t$$

$$\Rightarrow 2t = \pi/6$$



t= π/12 this is the time taken by oscillator to move from centre to midway

Now Time taken by oscillator to move from centre to extreme position is T/4

As T = 2π /ω thus

Time taken to reach to extreme position from the centre =

$$\frac{T}{4} = \frac{2\pi}{4\omega} = \frac{2\pi}{4\times 2} = \frac{\pi}{4}$$

Now time taken to reach oscillator to reach middle from extreme point is =

$$\frac{\pi}{4} - \frac{\pi}{12} = \frac{\pi}{6}\ sec$$

Q) A point moving in a straight line SHM has velocities v_1 and v_2 when its displacements from the mean position are x_1 and x_2 respectively. Show that the time period is

$$2\pi\sqrt{\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}}$$

<u>Solution</u>

From the formula for velocity

$$v(t) = \pm\, \omega\, (\sqrt{A^2 - x^2})$$

$$v_1^2 = \omega^2(A^2 - x_1^2)$$

$$A^2 = \frac{v_1^2}{\omega^2} + x_1^2 \quad ---eq(1)$$

$$A^2 = \frac{v_2^2}{\omega^2} + x_2^2 \quad ---eq(2)$$

From equation (1) and (2)

$$\frac{v_1^2}{\omega^2} + x_1^2 = \frac{v_2^2}{\omega^2} + x_2^2$$

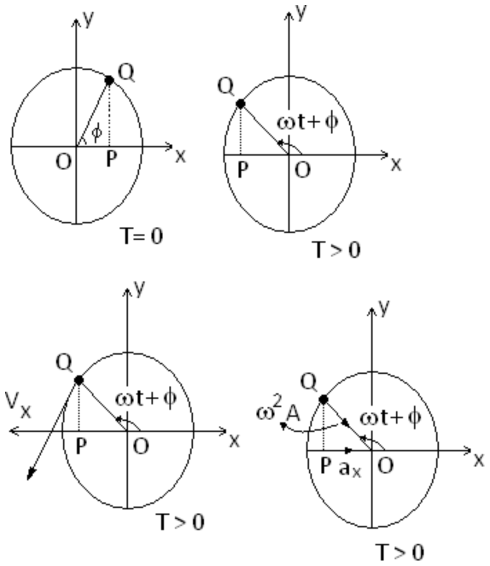
$$\frac{v_2^2}{\omega^2} - \frac{v_1^2}{\omega^2} = x_1^2 - x_2^2$$

$$\frac{1}{\omega^2}(v_2^2 - v_1^2) = x_1^2 - x_2^2$$

$$\frac{T^2}{(2\pi)^2}(v_2^2 - v_1^2) = x_1^2 - x_2^2$$

$$T = 2\pi\sqrt{\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}}$$

Relation between simple harmonic motion and uniform circular motion



In figure shown Q is a point moving on a circle of radius A with constant angular speed ω (in rad/sec). P is the perpendicular projection of Q on the horizontal diameter, along the x-axis.
Let us take Q as the reference point and the circle on which it moves the reference circle. As the reference point revolves, the projected point P moves back and forth along the horizontal diameter

Let the angle between the radius OQ and the x-axis at the time t=0 be called ϕ. At any time t later the angle between OQ and the x-axis is (ωt+ ϕ), the point Q moving with constant angular speed ω. The x-coordinate of Q at any time is, therefore

$X = A\cos(\omega t + \phi)$

i.e. P moves with simple harmonic motion

Thus, when a particle moves with uniform circular motion, its projection on a diameter moves with simple harmonic motion. The angular frequency ω of simple harmonic motion is the same as the angular speed of the reference point.

The velocity of Q is $v = \omega A$. The component of v along the x-axis is

$V_x = -v\sin(\omega t + \phi)$

$V_x = -\omega A\sin(\omega t + \phi)$,

Which is also the velocity of p. The acceleration of Q is centripetal and has a magnitude,

$a = \omega^2 A$

The component of 'a' along the x-axis is

$A_x = -a\cos(\omega t + \phi)$,

$A_x = -\omega^2 A\cos(\omega t + \phi)$,

Which is the acceleration of P

The force law for simple harmonic motion

We know that

$F = ma$

As $a = -\omega^2 x(t)$

$F = -m\omega^2 x(t)$

This force is restoring force

According to Hook's law, the restoring force is given by

$F = -kx(t)$

With k as spring constant

Thus $k = m\,\omega^2$

∴ angular frequency

$$\omega = \sqrt{\frac{k}{m}}$$

And frequency of oscillation

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

In many cases, the simple harmonic motion can also occur even without spring. In that case k is called the force constant of SHM and it is restoring force per unit displacement (K = -F/x)

Solved numerical

Q) A spring balance has a scale that reads 50kg. The length of the scale is 20cm. A body suspended from this spring, when displaced and released, oscillates with period of 0.6s. Find the weight of the body

Solution:

Here m = 50kg

Maximum extension of spring x = 20- 0 = 20 cm = 0.2 m

Periodic time T = 0.6s

Maximum force F = mg

F = 50× 9.8 = 490 N

K = F/x = 490/0.2 = 2450 N m^{-1}

As

$$T = 2\pi\sqrt{\frac{m}{k}}$$

$$m = \frac{(0.6)^2 \times 2450}{4 \times (3.14)^2} = 22.36\ kg$$

∴ Weight of the body = mg = 22.36 ×9.8 = 219.1 N = 22.36 kgf

Examples of simple harmonic motion

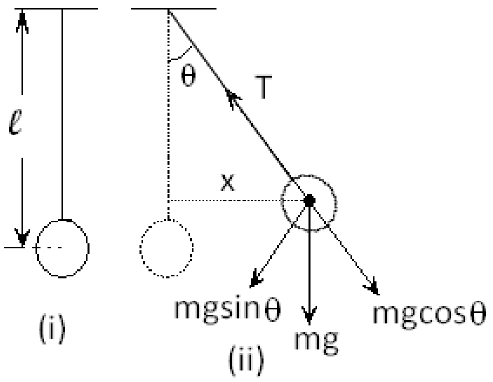
Simple pendulum

A simple pendulum consists of a heavy particle suspended from a fixed support through a light, inextensible and torsion less string

The time period of simple pendulum can be found by force or torque method and also by energy method

(a) Force method: the mean position or the equilibrium position of the simple pendulum is when θ = 0 as shown in figure(i). the length of the string is l, and mass of the bob is m

When the bob is displaced through distance 'x', the forces acting on it are shown in figure(ii)



The restoring force acting on the bob to bring it to the mean position is

F = - mgsinθ (-ve sign indicates that force is directed towards the mean position)

For small angular displacement

Sinθ ≈θ = x/l

$$\therefore F = -mg\frac{x}{l}$$

$$\therefore a = -g\frac{x}{l}$$

Comparing it with equation of simple harmonic motion a = $-\omega^2 x$

$$\omega^2 = \frac{g}{l}$$

$$\therefore \omega = \sqrt{\frac{g}{l}}$$

Time period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$$

(b) Torque method:

Now taking moment of force acting on the bob about O

$$\tau = -(mgsin\theta)l - - - eq(1)$$

Also from Newton's second law

$$\tau = I\alpha - - - eq(2)$$

From equation(1) and (2)

$$I\alpha = -(mgsin\theta)l$$

Since θ is small

$$I\alpha = -(mg\theta)l$$

But Moment of inertia I = ml^2

$$ml^2\alpha = -(mg\theta)l$$

$$\therefore \alpha = -\left(\frac{g}{l}\right)\theta$$

Comparing with simple harmonic motion equation ,$\alpha = -\omega^2\theta$

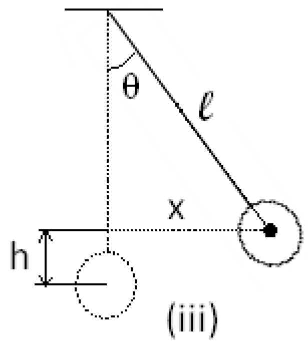
$$\omega = \sqrt{\frac{g}{l}}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$$

Note: component of force $mg\cos\theta$ cannot produce torque because it passes through fixed point

(c) Energy method:

Let the potential energy at the mean position be zero. Let the bob is displaced through an angle 'θ'. Let its velocity be 'v'



(iii)

Then potential energy at the new position

$U = mgl(1-\cos\theta)$

Kinetic energy at this instant $K = (1/2)\, mv^2$

Total mechanical energy at this instant

$E = U + K$

$E = mgl(1-\cos\theta) + (1/2)\, mv^2$

We know, in simple harmonic motion E = constant

$$\frac{dE}{dt} = 0$$

$$\Rightarrow mgl\left[\sin\theta\frac{d\theta}{dt}\right] + mv\frac{dv}{dt} = 0$$

Bur $v = \omega l$

$$\therefore v = l\frac{d\theta}{dt}$$

$$\therefore mgl\left[\sin\theta\frac{d\theta}{dt}\right] + ml\frac{d\theta}{dt}\frac{dv}{dt} = 0$$

$$\therefore g[\sin\theta] + \frac{dv}{dt} = 0$$

$$\frac{dv}{dt} = a = -g[\sin\theta] \approx g\theta$$

But $\theta = x/l$

$$\therefore a = -g\frac{x}{l}$$

But $a=-\omega^2 x$

$$\therefore \omega^2 = \frac{g}{l}$$

$$\omega = \sqrt{\frac{g}{l}}$$

Periodic time

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$$

(a) Simple pendulum in a lift:

If the simple pendulum is oscillating in a lift moving with acceleration a, then the effective g of the pendulum is

$g_{eff} = g \pm a$

+ sign is taken when lift is moving upward

- ve sign is taken when lift is moving downward

Hence periodic time of pendulum

$$T = 2\pi\sqrt{\frac{l}{g \pm a}}$$

(b) Simple pendulum in the compartment of a train

If the simple pendulum is oscillating in a compartment of a train accelerating or retarding horizontally at the rate 'a' then the effective value of g is

$$g_{eff} = \sqrt{g^2 + a^2}$$

Hence periodic time of pendulum

$$T = 2\pi\sqrt{\frac{l}{\sqrt{g^2 + a^2}}}$$

(c) Seconds pendulum

The pendulum having the time-period of two seconds, is called the second pendulum. It takes one second to go from one end to the other end during oscillation. It also crosses the mean position at every one second

Solved Numerical

Q) Length of a second's pendulum on the surface of earth is l_1 and l_2 at height 'h' from the surface of earth. Prove that the radius of the earth is given by

$$R_e = \frac{h\sqrt{l_2}}{\sqrt{l_1} - \sqrt{l_2}}$$

Solution:

Period of second's pendulum is 2sec

$$2 = 2\pi\sqrt{\frac{l_1}{g}}$$

And g at surface of earth is given by

$$g = \frac{GM_e}{R_e^2}$$

$$\therefore\ 2 = 2\pi\sqrt{\frac{l_1 R_e^2}{GM_e}} \quad ---eq(1)$$

g at height h from surface of earth is

$$g = \frac{GM_e}{(R_e + h)^2}$$

Thus at height h

$$2 = 2\pi\sqrt{\frac{l_2(R_e + h)^2}{GM_e}} \quad ---eq(2)$$

From equation(1) and (2) we get

$$2\pi\sqrt{\frac{l_2(R_e + h)^2}{GM_e}} = 2\pi\sqrt{\frac{l_1 R_e^2}{GM_e}}$$

$$l_2(R_e + h)^2 = l_1 R_e^2$$

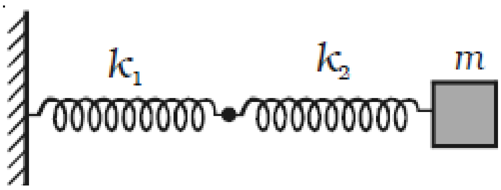
$$\sqrt{l_2}\,(R_e + h) = \sqrt{l_1}\,R_e$$

$$\sqrt{l_2}\,h = R_e(\sqrt{l_1} - \sqrt{l_2})$$

$$R_e = \frac{h\sqrt{l_2}}{\sqrt{l_1} - \sqrt{l_2}}$$

Combinations of springs

Series combination



As show in figure consider a series combination of two massless spring of spring constant k_1 and k_2

In this system when the combination of two springs is displaced to a distance y, it produces extension x_1 and x_2 in two springs of force constants k_1 and k_2.
$F = -k_1 x_1$; $F = -k_2 x_2$

where F is the restoring force.

$$x = x_1 + x_2 = -F \left[\frac{1}{k_1} + \frac{1}{k_2}\right]$$

We know that $F = -kx$
$x = -F/k$
From the above equations,

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$$

$$k = \frac{k_1 k_2}{k_1 + k_2}$$

Time period T

$$T = 2\pi\sqrt{\frac{m(k_1 + k_2)}{k_1 k_2}}$$

Frequency f

$$f = \frac{1}{2\pi}\sqrt{\frac{k_1 k_2}{m(k_1 + k_2)}}$$

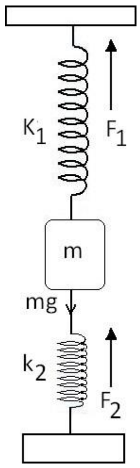
If both the springs have the same spring constant,
$k_1 = k_2 = k$. then equivalent spring constant k' = k/2

$$T = 2\pi\sqrt{\frac{m(2)}{k}}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k'}{m}} = \frac{1}{2\pi}\sqrt{\frac{k}{2m}}$$

Parallel combination

(i) Consider a situation as shown in figure where a body of mass m is attached in between the two massless springs of spring constant k_1 and k_2. let the body is left free for SHM in vertical plane after pulling mass m In this situation when a body is pulled lower through small displacement y. lower spring gets compressed by y, while upper spring elongate by y. hence restoring forces F_1 and F_2 set up in both these springs will act in the same direction.



Net restoring force will be

$F = F_1 + F_2$

$F = - k_1y - k_2y$

$F= - (k_1 + k_2)\ y$

If k' is the is equivalent spring constant then

$F = -k'y$ thus

$k' = k_1 + k_2$

Now periodic time T

$$T = 2\pi\sqrt{\frac{m}{k'}} = \sqrt{\frac{m}{k_1 + k_2}}$$

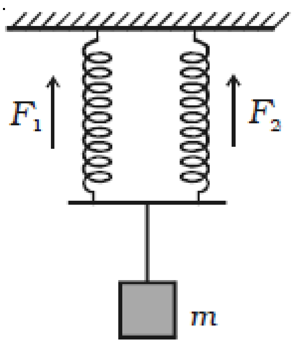
Frequency

$$f = \frac{1}{2\pi}\sqrt{\frac{k'}{m}} = \frac{1}{2\pi}\sqrt{\frac{k_1 + k_2}{m}}$$

If $k_1 = k_2 = k$ then

$$T = 2\pi\sqrt{\frac{m}{2k}}$$

(ii) Two massless springs of equal lengths having force constants k_1 and k_2 respectively are suspended vertically from a rigid support as shown in figure. At their free ends, a block of mass m having non-uniform density distribution is suspended so that spring undergoes equal extension



In this situation two bodies are pulled down through a small distance y and the system is made to perform SHM in vertical plane

Here, the springs have different force constants. Moreover the increase in their length is same. Therefore, the load is distributed equally between the springs. Hence, the restoring force developed in each spring is different.

If F_1 and F_2 are the restoring forces set up due to extension of springs, then

$F_1 = -k_1\, y$ and $F_2 = \ k_2y$

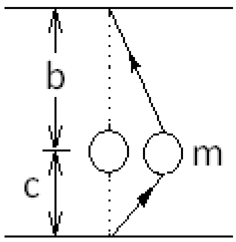
Also the total restoring force

$F = F_1 + F_2$

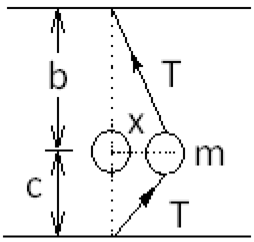
$F = F_1 + F_2$

$F = - k_1y - k_2y$
$F= - (k_1 + k_2) y$
If k' is the is equivalent spring constant then
$F = -k'y$ thus
$k' = k_1 + k_2$
Frequency and periodic time will be same as given in (i)

Q) A small mass m is fastened to a vertical wire, which is under tension T. What will be the natural frequency of vibration of the mass if it is displaced laterally a slight distance and then released?



Solution



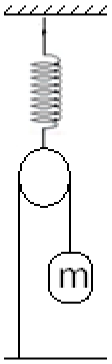
Sphere is displaced by a very small distance x thus angle formed is also small
Using trigonometry we can find Restoring force $F_b = T\sin\theta = T(x/b)$
Now restoring force per unit displacement $k_1 = F_b /x = T/b$
$F_c = T\sin\theta = T(x/c)$
Now restoring force per unit displacement $k_2 = F_c /x = T/c$
Since both restoring forces are parallel combination

$$f = \frac{1}{2\pi}\sqrt{\frac{\frac{T}{b}+\frac{T}{c}}{m}}$$

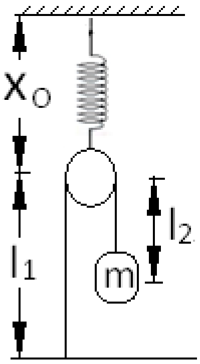
$$f = \frac{1}{2\pi}\sqrt{\frac{T}{m}\left(\frac{1}{b}+\frac{1}{c}\right)}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{T}{m}\left(\frac{b+c}{bc}\right)}$$

Q) The spring has a force constant k. the pulley is light and smooth while the spring and the string are light. If the block of mass 'm' is slightly displaced vertically and released, find the period of vertical oscillation



Solution: When mass m is pulled by distance of x, increase in length of spring is x/2 as explained below



$x_0 + l_1$ = constant
dx_0 be the increase in length of spring
$dx_0 + dl_1 = 0$
and $dx_0 + dl_2 = x$
Thus $2dx_0 + dl_1 + dl_2 = x$
As string is inextensible $dl_1 + dl_2 = 0$
Thus $2dx_0 = x$ or $dx_0 = x/2$

Before pulling mass m spring was in equilibrium
$2T_0 = kx_0$
And $T_0 = mg$
Thus $2mg = kx_0$
When spring stretch by x/2 then tension in spring is T

$$F = k\left(x_0 + \frac{x}{2}\right)$$

$$\text{Or } 2T = k\left(x_0 + \frac{x}{2}\right)$$

$$2T = kx_0 + \frac{kx}{2}$$

$$2T = 2T_0 + \frac{kx}{2}$$

$$2T - 2T_0 = \frac{kx}{2}$$

$$T - T_0 = \frac{kx}{4}$$

Restoring force on mass m is $T-T_0$ which is proportional to displacement

Thus restoring force constant k' = k/4

Period of oscillation

$$T = 2\pi\sqrt{\frac{m}{k'}} = 2\pi\sqrt{\frac{m}{\frac{k}{4}}} = 4\pi\sqrt{\frac{m}{k}}$$

Differential equation of simple harmonic motion

According to Newton's second law of motion

$$F = ma = m\frac{d^2y(t)}{dt^2}$$

Comparing this with F = -ky(t)

$$m\frac{d^2y(t)}{dt^2} = -ky(t)$$

$$\frac{d^2y(t)}{dt^2} = -\frac{k}{m}y(t)$$

$$\frac{d^2y(t)}{dt^2} = -\omega^2 y(t)$$

$$\frac{d^2y(t)}{dt^2} + \omega^2 y(t) = 0$$

This is the second order differential equation of the simple harmonic motion. The solution of this equation is of the type

Y(t) = A sinωt or y(t)=Bcosωt

Or any linear combination of sine and cosine function

Y(t) = A sinωt + Bcosωt

Solved Numerical

Q) The SHM is represented by y = 3sin314t + 4cos314t. y in cm and in t in second. Find the amplitude, epoch the periodic time and the maximum velocity of SHO

Solution

Y = A sin(ωt + φ)

Y = A cosφsin ωt + Asinφcosωt

Comparing with 3sin314t + 4cos314t

3 = Acosφ and 4 = Asinφ

$\therefore A^2\cos^2\varphi + A^2\sin^2\varphi = 3^2 + 4^2$

$A^2 = 25$

A = 5 cm

The initial phase (epoch) is obtained as

$$tan\varphi = \frac{sin\varphi}{cos\varphi} = \frac{4}{3}$$

$$\varphi = tan^{-1}\left(\frac{4}{3}\right)$$

$$\varphi = 53^o8'$$

Now

$$T = \frac{2\pi}{\omega\pi} = \frac{2}{314} = 0.02\ s$$

Maximum velocity

$V_{max} = \omega A = 314 \times 5 = 1570$ cm/s

Q) The vertical motion of a ship at sea is described by the equation $\frac{d^2x}{dt^2} = -4x$ where x in metre is the vertical height of the ship above its mean position. If it oscillates through a total distance of 1m in half oscillation, find the greatest vertical speed and the greatest vertical acceleration.

Solution

Comparing given with standard equation for oscillation we get

$\omega^2 = 4$ or $\omega = 2$

given amplitude A = 1/2

$V_{max} = \omega A = 2\times (1/2) = 1$m/s

$A_{max} = \omega^2 A = 2^2 \times (1/2) = 2$ m/s^2

Total Mechanical Energy in Simple Harmonic Oscillator

The total energy (*E*) of an oscillating particle is equal to the sum of its kinetic energy and potential energy if conservative force acts on it.

Kinetic energy

Kinetic energy of the particle of mass *m* is

$$K = \frac{1}{2} mv^2 = \frac{1}{2} m\omega^2(A^2 - x^2) - - - eq(1)$$

Potential energy

From definition of SHM $F = -kx$ the work done by the force during the small displacement dx is $dW = -F.dx = -(-kx)\ dx = kx\ dx$

∴ Total work done for the displacement *x* is,

$$W = \int dw = \int_0^x kx\ dx$$

$k=\omega^2 m$

$$W = \int_0^x m\omega^2 x\, dx = \frac{1}{2} m\omega^2 x^2$$

This work done is stored in the body as potential energy

$$U = \frac{1}{2} m\omega^2 x^2 \quad --- (2)$$

Total energy *E* = *K* + *U*

$$E = \frac{1}{2} m\omega^2 (A^2 - x^2) + \frac{1}{2} m\omega^2 x^2$$

$$E = \frac{1}{2} m\omega^2 A^2$$

Special cases

(i) When the particle is at the mean position *x*= 0, from eqn (1) it is known that kinetic energy is maximum total energy is wholly kinetic $K_{max} = \frac{1}{2} m\omega^2 A^2$

and from eqn. (2) it is known that potential energy is zero.

(ii) When the particle is at the extreme position y = *+a*, from eqn. (1) it is known that kinetic energy is zero and from eqn. (2) it is known that Potential energy is maximum. Hence the total energy is wholly potential. $U_{max} = \frac{1}{2} m\omega^2 A^2$

(iii) When y = A/2

$$K = \frac{1}{2} m\omega^2 \left[A^2 - \left(\frac{A}{2}\right)^2\right]$$

$$K = \frac{3}{4}\left(\frac{1}{2} m\omega^2 A^2\right)$$

$$K = \frac{3}{4}(E)$$

$$U = \frac{1}{2} m\omega^2 \left(\frac{A}{2}\right)^2$$

$$U = \frac{1}{4}\left(\frac{1}{2} m\omega^2 A^2\right)$$

If the displacement is half of the amplitude K and U are in the ratio 3 : 1,

Solved numerical

Q) If a particle of mass 0.2kg executes SHM of amplitude 2cm and period of 6 sec find(i0the total mechanical energy at any instant (ii) kinetic energy and potential energies when the displacement is 1cm

Solution:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{6}$$

(i) Total mechanical energy at any instant is given by

$$E = \frac{1}{2} m\omega^2 A^2$$

$$E = \frac{1}{2}(0.2)\left(\frac{2\pi}{6}\right)^2 (2 \times 10^{-2})^2$$

$$E = 0.1 \times \frac{4\pi^2}{36} \times 4 \times 10^{-4}$$

$$E = 4.39 \times 10^{-5} J$$

(ii) K.E. at the instant when displacement x is given by

$$K = \frac{1}{2} m\omega^2 (A^2 - x^2)$$

$$K = \frac{1}{2}(0.2)\left(\frac{2\pi}{6}\right)^2 (4 \times 10^{-4} - 1 \times 10^{-4})\, J$$

$$K = 3.29 \times 10^{-5} J$$

P.E. energy at that instant = Total energy – K.E

= (4.39 -3.29) $\times 10^{-5}$

=1.1$\times 10^{-5}$J

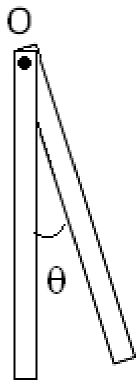
Angular Simple Harmonic Motion

A body to rotate about a given axis can make angular oscillations. For example, a wooden stick nailed to a wall can oscillate about its mean position in the vertical plane

The conditions for an angular oscillation to be angular harmonic motion are

(i) When a body is displaced through an angle from the mean position, the resultant torque is proportional to the angle displaced

(ii) This torque is restoring in nature and it tries to bring the body towards the mean position



If the angular displacement of the body at an instant is θ , then resultant torque on the body

τ = -kθ

if the momentum of inertia is I, the angular acceleration is

$$\alpha = \frac{\tau}{I} = -\frac{k}{I}\theta$$

Or

$$\frac{d^2\theta}{dt^2} = -\omega^2 \theta \quad --- eq(1)$$

Here

$$\omega = \sqrt{\frac{k}{I}}$$

Solution of equation (1) is

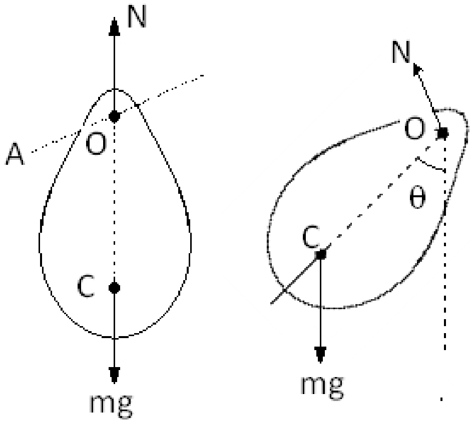
$$\theta = \theta_0 \sin(\omega t + \varphi)$$

Where θ_0 is the maximum angular displacement on either side. Angular velocity at time 't' is given by

$$\omega = \frac{d\theta}{dt} = \theta_0 \omega cos(\omega t + \varphi)$$

Physical pendulum

An rigid body suspended from a fixed support constitutes a physical pendulum.



As shown in figure is a physical pendulum. A rigid body is suspended through a hole at O. When the centre of mass C is vertically below O at a distance of 'l', the body may remain at rest.

The body is rotated through an angle θ about a horizontal axis OA passing through O and perpendicular to the plane of motion The torque of the forces acting on the body, about the axis OA is τ = mglsinθ, here l = OC

If momentum of inertia of the body about OA is I, the angular acceleration becomes

$$\alpha = \frac{\tau}{I} = -\frac{mgl}{I} sin\theta$$

For small angular displacement sinθ = θ

$$\alpha = -\left(\frac{mgl}{I}\right)\theta$$

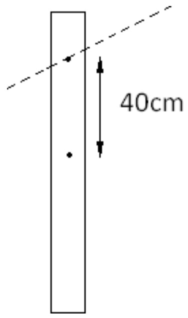
Comparing with $\alpha = -\omega^2\theta$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgl}}$$

Solved Numerical

Q) A uniform meter stick is suspended through a small-hole at the 10cm mark. Find the time period of small oscillations about the point of suspension

Solution:



Let the mass of stick be M. The moment of inertia of the stick about the axis of rotation through the point of suspension is

$$I = \frac{Ml^2}{12} + Md^2 = M\left(\frac{l^2}{12} + d^2\right)$$

Centre of mass of the stick is at 50 cm from top thus it is at distance 40 cm from the small hole thus d = 40 cm, given length of stick = 1m

Time period

$$T = 2\pi\sqrt{\frac{I}{Mgd}}$$

$$T = 2\pi\sqrt{\frac{M\left(\frac{l^2}{12} + d^2\right)}{Mgd}}$$

$$T = 2\pi\sqrt{\frac{\left(\frac{l^2}{12} + d^2\right)}{gd}}$$

$$T = 2 \times 3.14\sqrt{\frac{\frac{1}{12} + (0.4)^2}{9.8 \times 0.4}}$$

T=1.56 sec

Torsional pendulum

In torsional pendulum, an extended body is suspended by a light thread or wire. The body is rotated through an angle about the wire as the axis of rotation.

The wire remains vertical during this motion but a twist 'θ' is produced in the wire. The twisted wire exerts a restoring torque on the body, which is proportional to the angle of the twist.

$\tau \propto -\theta$; $\tau = -k\theta$; k is proportionality constant and is called torsional constant of the wire. If l be the moment of inertia of the body about vertical axis, the angular acceleration is

$$\alpha = \frac{\tau}{I} = \frac{-k}{I}\theta = -\omega^2\theta$$

$$\omega = \sqrt{\frac{k}{I}}$$

Time period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{k}}$

Q) The moment of inertia of the disc used in a torsional pendulum about the suspension wire is 0.2 kg-m^2. It oscillates with a period of 2s. Another disc is placed over the first one and the time period of the system becomes 2.5 s. Find the moment of inertia of the second disc about the wire

Solution

Let the torsional constant of the wire be k

$$T = 2\pi\sqrt{\frac{I}{k}}$$

$$2 = 2\pi\sqrt{\frac{0.2}{k}} \quad --- eq(1)$$

When a second disc having moment of inertia I_1 about the wire is added, the time period is

$$2.5 = 2\pi\sqrt{\frac{0.2 + I_1}{k}} \quad --- eq(2)$$

From eq(1) and (2)

$$I_1 = 0.11 \text{ kg-m}^2$$

Two Body System

In a two body oscillations, such as shown in the figure, a spring connects two objects, each of which is free to move. When the objects are displaced and released, they both oscillate. The relative separation $x_1 - x_2$ gives the length of the spring at any time. Suppose its unscratched length is L; then $x = (x_1 - x_2) - L$ is the change in length of the spring, and $F = kx$ is the magnitude of the force exerted on each particle by the spring as shown in figure.

Applying Newton's second law separately to the two particles, taking force component along the x-axis, we get

$$m_1 \frac{d^2x_1}{dt^2} = -kx \; and \; m_2 \frac{d^2x_2}{dt^2} = +kx$$

Multiplying the first of these equations by m_2 and the second by m_1 and then subtracting,

$$m_1m_2\frac{d^2x_1}{dt^2} - m_1m_2\frac{d^2x_2}{dt^2} = -m_2kx - m_1kx$$

This can be written as

$$\frac{m_1m_2}{(m_1+m_2)}\frac{d^2}{dt^2}(x_1 - x_2) = -kx$$

$$\mu\frac{d^2}{dt^2}(x_1 - x_2) = -kx \quad --- eq(1)$$

Here μ is known as reduced mass and has dimension of mass

$$\mu = \frac{m_1m_2}{(m_1+m_2)}$$

Since L is constant

$$\frac{d^2}{dt^2}(x_1 - x_2) = \frac{d^2}{dt^2}(x + L) = \frac{dx}{dt}$$

Equation (i) becomes

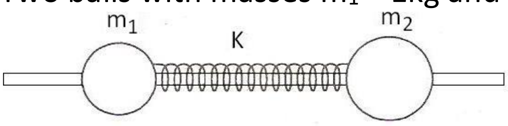
$$\mu\frac{d^2x}{dt^2} = -kx$$

$$\frac{d^2x}{dt^2} = -\frac{k}{\mu}x$$

Thus periodic time

$$T = 2\pi\sqrt{\frac{\mu}{k}}$$

Q) Two balls with masses m_1 = 1kg and m_2 = 2kgkg are slipped on a thin smooth horizontal



rod. The balls are interconnected by a light spring of spring constant 24 N/m. the left hand ball is imparted the initial velocity v_1= 12 cm/s.

Find (a) the oscillation frequency of the system (b) the energy and amplitude of oscillation

Solution

Reduced mass

$$\mu = \frac{m_1m_2}{(m_1 + m_2)} = \frac{1 \times 2}{1 + 2} = \frac{2}{3}\ kg$$

Frequency

f = 1/T

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{\mu}}$$

$$f = \frac{1}{2 \times 3.14}\sqrt{\frac{24 \times 3}{2}} = 0.955\ sec$$

Initial velocity given to mass m_1 is v_1

For undamped oscillation, this initial energy remains constant

Hence total energy of S.H.M. of two balls is given as

$$E = \frac{1}{2}\mu v_1^2$$

If amplitude of oscillation is A then

$$\frac{1}{2}\mu v_1^2 = \frac{1}{2}kA^2$$

$$A = \sqrt{\frac{\mu}{k}}v_1$$

$$A = \sqrt{\frac{2}{3 \times 24}} \times 0.12 = 0.02\ m$$

Or A = 2 cm

Damped oscillation

Experimental studies showed that the resistive force acting on the oscillator in a fluid medium depends upon the velocity of the oscillator

Thus resistive force or damping force acting on the oscillator is

$F_d \propto v$

$\therefore F_d = -nv$

Here b is damping constant and has SI units kg/second. The negative sign indicates that the force F_d opposes the motion

Thus, a damped oscillator oscillate under the influence of the following forces

(i)Restoring force $F_x = -kx$ and

(ii) Resistive force $F_d = -bv$

Net force $F = F_x + F_d$

According to second law of motion

ma = -ky – bv

$$m\frac{d^2x}{dt^2} = -\mathrm{ky} - \mathrm{b}\frac{\mathrm{dx}}{\mathrm{dt}}$$

$$\frac{d^2x}{dt^2} + \frac{\mathrm{b}}{\mathrm{m}}\frac{\mathrm{dx}}{\mathrm{dt}} + \frac{\mathrm{k}}{\mathrm{m}}\mathrm{y} = 0 \quad --- \mathrm{eq}(1)$$

Solution of this equation is

$$x(t) = A\,e^{-bt/2m}\,sin(\omega' t + \varphi)$$

Here $A\,e^{-bt/2m}$ is the amplitude of the damped oscillation and decreases exponentially with time

The angular frequency ω' of the damped oscillator is given by

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

Energy of oscillator

$$E = \frac{1}{2}kA^2 e^{-bt/m}$$

Above equation is valid only if b << √(km)

Natural Oscillations

When a system capable of oscillating is given some initial displacement from its equilibrium position and left free (i.e. in absence of any external force) it begins to oscillate. Thus the oscillations performed by it in absence of any resistive forces are known as natural oscillations. The frequency of natural oscillations is known as natural frequency f_0 and corresponding angular frequency is denoted by ω^0

Forced Oscillation

Oscillations of the system under the influence of an external periodic force are forced oscillation

Consider an external periodic force F = F_0 sinωt acting on the system which is capable to oscillate

Equation for oscillation can be witten as

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + ky = F_0 \sin\omega t$$

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}y = \frac{F_0}{m}\sin\omega t$$

The solution of equation is given by

X = A sin(ωt + φ)

Here, A and φ are the constants of the solution they are found as,

$$A = \frac{F_0}{[m^2(\omega_0^2 - \omega^2)^2 + b^2\omega^2]^{1/2}}$$

$$\varphi = tan^{-1}\frac{\omega x}{v_0}$$

Here m is the mass of oscillator, v_0 is velocity of oscillator, x is the displacement of oscillator.

(i) for small damping factor

$$m(\omega_0^2 - \omega^2) \gg b\omega$$

Equation for amplitude becomes

$$A = \frac{F_0}{m^2(\omega_0^2 - \omega^2)^2}$$

(ii) For large damping factor

$$b\omega \gg m(\omega_0^2 - \omega^2)$$

$$A = \frac{F_0}{b\omega}$$

If when value of ω approaches ω_0 the amplitude becomes maximum. This phenomenon is known as resonance. The value of ω for which resonance occurs is known as the resonant frequency.

Solved numerical

Calculate the time during which the amplitude becomes $A/2^n$ in case of damped oscillations, where A = initial amplitude

Solution:

$$A(t) = Ae^{-bt/2m}$$

But $A(t)= A/2^n$

$$\therefore \frac{A}{2^n} = Ae^{-bt/2m}$$

Taking log to the base e on both sides

$$\frac{bt}{2m} = n\ln 2$$

$$t = \frac{2mn\ln 2}{bt} = \frac{2mn}{bt}(0.693)$$

RAY OPTICS AND OPTICAL INSTRUMENTS

SECTION I

REFLECTION OF LIGHT

Nature of light

- Light is an electromagnetic radiation which causes sensation in eyes.
- Wavelength of visible light is 400 nm to 750 nm
- Speed of light in vacuum is highest speed attainable in nature 3.0×10^8 m/s
- Wavelength of light is very small compared to the size of the ordinary objects, thus light wave is considered to travel from one point to another along straight line joining two points.
- The straight path joining two points is called ray of light.
- Bundle of rays is called a beam of light
- Light show optical phenomenon such as reflection , refraction, interference and diffraction

REFLECTION OF LIGHT BY SPHERICAL MIRRORS

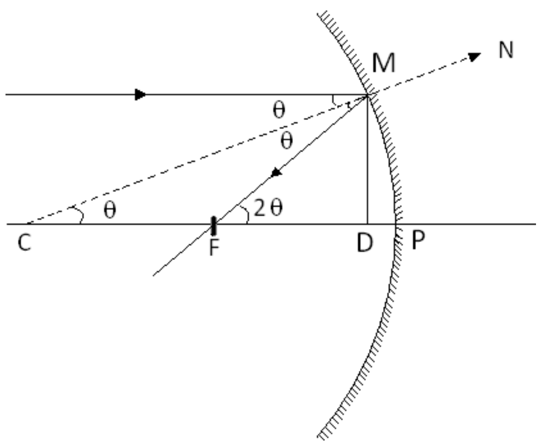
Law of reflection. I) The angle of incidence (angle between incident ray and normal to the surface) and angle of reflection (angle between reflected ray and normal to the surface) are equal.

ii) Incident ray, reflected ray and normal to the reflecting surface at the point of incidence lie in the same plane.

Note : Normal to the curved surface always passes through the centre of curvature

Sign convention: (i) All distances are measured from the pole of the mirror.
(ii) The distance in the direction of incidence light is taken as positive.
(iii) Distance in opposite to the direction of incident light is taken as negative
(iv) Distance above the principal axis is taken as positive.
(v) Distance below the principal axis taken as negative.

FOCAL LENGTH OF SPHERICAL MIRROR



Let P the pole of concave mirror, F be focal point and C be the centre of curvature.
Consider incident light parallel to principal axis strikes the mirror at point M and reflected rays passes through focal point F.
MD is perpendicular from M on principal axis.
Let ∠MCP = θ
Then from geometry of figure, ∠MFP = 2θ.
Now $\tan\theta = \frac{MD}{CD}$ and $\tan 2\theta = \frac{MD}{FD}$ –eq(1)

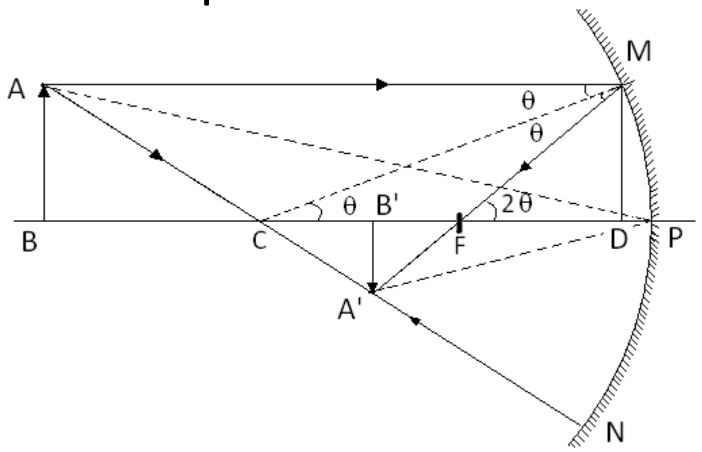
For small θ , tan θ = θ and tan2 θ = 2 θ

Therefore from eq(1) $2\frac{\mathrm{MD}}{\mathrm{CD}} = \frac{\mathrm{MD}}{\mathrm{FD}}$

Thus FD = CD/2 But CD = R and FD = f

Thus f = R/2

The Mirror equation



As shown in figure AB is object while A'B' is image of the object . AM and AN are two incident rays emitted from point A. Let F be focal point and FD = f focal length

For small aperture , DP will be very small neglecting DP we will take FD = FP = f

I) In Δ A'B'F and ΔMDF are similar as ∠MDF = ∠ A'B'F

$$\frac{A'B'}{MD} = \frac{B'F}{FD}$$

As MD = AB and FD = FP

$$\frac{A'B'}{AB} = \frac{B'F}{FP}$$

ii) In Δ A'B'P and Δ ABP are also similar. Therefore

$$\frac{A'B'}{AB} = \frac{PB'}{PB}$$

Thus comparing above two equations

$$\frac{B'F}{FP} = \frac{PB'}{PB}$$

But B'F = PB'-PF

$$\frac{PB' - PF}{FP} = \frac{PB'}{PB} \quad - eq(1)$$

From sign convention

Focal length = PF = -f

Image distance PB' = -v

Object distance = PB = -u

Putting the values in equation (1)

$$\frac{-v - (-f)}{-f} = \frac{-v}{-u}$$

$$\frac{v - f}{f} = \frac{v}{u}$$

$$\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$$

This relation is known as mirror equation

Magnification

In Δ A'B'P and Δ ABP are also similar. Therefore

$$\frac{A'B'}{AB}=\frac{PB'}{PB}$$

From sign convention

PB' = image distance =-v

PB = object distance = -u

A'B' = size of image = -h '

AB = size of object = h

$$\frac{-h'}{h}=\frac{-v}{-u}$$

$$\frac{h'}{h}=-\frac{v}{u}$$

Magnification, m = size of the image / size of object = h'/h

$$m=-\frac{v}{u}$$

Note : If m is negative , image is real and inverted

If m is positive image is virtual and inverted

If |m| = 1 , size of the object = size of image

If |m| >1 size of image > size of the object

If |m| < 1 size of the image < size of the object

Solved Numerical

Q) An object is placed in front of concave mirror at a distance of 7.5 cm from it. If the real image is formed at a distance of 30 cm from the mirror, find the focal length of the mirror. What should be the focal length if the image is virtual?

Solution: Case I: When the image is real

U = -7.5 cm; v=-30cm; f= ?

We know that

$$\frac{1}{f}=\frac{1}{v}+\frac{1}{u}$$

$$\frac{1}{f}=\frac{1}{-30}+\frac{1}{-7.5}$$

$$\frac{1}{f} = \frac{-5}{30}$$

f = -6 cm

The negative sign shows that the spherical mirror is convergent or concave

Case II : When image is virtual

u = -7.5 cm ; v = +30 cm

We know

$$\frac{1}{f} = \frac{1}{v} + \frac{1}{u}$$

$$\frac{1}{f} = \frac{1}{30} + \frac{1}{-7.5}$$

$$\frac{1}{f} = \frac{-3}{30} = \frac{-1}{10}$$

f = -10 cm

Q) An object 0.5 cm high is placed 30 cm from convex mirror whose focal length is 20 cm. Find the position, size and nature of the image.

Solution: We have

U = -30 cm, f = +20 cm

Form mirror formula

$$\frac{1}{20} = \frac{1}{v} + \frac{1}{-30}$$

$$\frac{1}{v} = \frac{1}{12}$$

V = 12 cm

The image is formed 12 cm behind the mirror. It is virtual and erect

m = h'/h = -v/u = -12/30

$$m = \frac{h'}{h} = \frac{-v}{u} = \frac{-12}{-30} = 0.4$$

h' = mh = 0.4 ×0.5 = 0.2 cm

positive sign of m indicate image is erect.

Q) A thin rod AB of length 10 cm is placed on the principal axis of a concave mirror such that its end B is at a distance of 40 cm from the mirror and end A is further away from the mirror. If the focal length of the mirror is 20cm, find the length of the image of rod

Solution: given f = -20 cm, distance of B = u_1 =- 40 cm,

Since B is at centre of curvature image will be formed at -40 cm

Distance of A u_2 = -50 cm

$$\frac{1}{-20} = \frac{1}{v} + \frac{1}{-50}$$

V = -33.3 cm

Image of A is also on the side of object , Now length of image = 40-33.3 = 6.70 cm

SECTION II
REFRACTION OF LIGHT

REFRACTION

When an obliquely incident ray of light travels from one transparent medium to other it changes it direction of propagation at the surface separating two medium. This phenomenon is called as refraction. This phenomenon is observed as light have different velocity in different medium.

Laws of refraction

(i) The incident ray, refracted ray and the normal to the interface at the point of incidence, all lie in the same plane

(ii) The ratio of the sine of the angle of incidence to the sine of angle of refraction is constant.

$$\frac{sini}{sinr} = n_{21}$$

Where n_{21} is constant, called refractive index of the second medium with respect to first medium. Equation is known as Snell's law of refraction

Case i) $n_{21} > 1$, $r < i$, refracted ray bends towards normal , then medium 2 is called optically denser medium

Case ii) $n_{21} < 1$, $r > i$, refracted ray bends away from normal, then medium 2 is called optically rarer medium

If n_{21} is refractive index of medium 2 with respect to medium 1 and n_{12} is the refractive index of medium 1 with respect to medium 2 then

$$n_{21} = \frac{1}{n_{12}}$$

Similarly

$$n_{32} = n_{31} \times n_{12}$$

General form of Snell's law

$$n_{21} = \frac{v_1}{v_2}$$

Absolute refractive index is given by formula

$$n = \frac{C}{v}$$

Thus $n_1 = C/v_1$ and $n_2 = C/v_2$

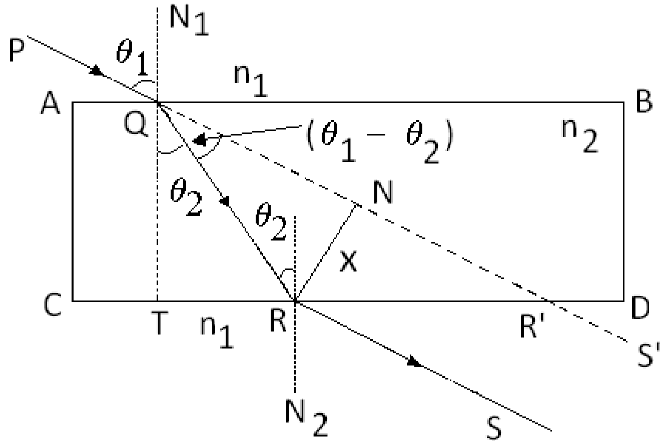
$v_1 = C/n_1$ and $v_2 = C/n_2$

$$n_{21} = \frac{C/n_1}{C/n_2} = \frac{n_2}{n^1}$$

$$\frac{sini}{sinr} = \frac{n_2}{n_1}$$

$n_1 sini = n_2 sinr$

Lateral shift



As shown in figure light ray undergoes refraction twice, once at top (AB) and then from bottom (CD) surfaces of given homogeneous medium.

The emergent ray is parallel to PQR'S' ray. Here PQR'S' is the path of light in absence of the other medium.

Since emergent ray is parallel to the incident ray but shifted sideways by distance RN. This RN distance is called lateral shift (x)

Calculation of lateral shift

Let n_1 and n_2 be the refractive indices of the rarer and denser medium, respectively. Also $n_1 < n_2$ From figure $\angle RQN = (\theta_1 - \theta_2)$, $RN = X$

i) From ΔQRN, $\sin(\theta_1 - \theta_2) = RN/QR = X/QR$ --eq(1)

ii) In ΔQTR, $\cos\theta_2 = QT/QR$

$\therefore QR = QT/\cos\theta_2$; $QR = t/\cos\theta_2$

From equation (1)

$$\sin(\theta_1 - \theta_2) = \frac{x}{\frac{t}{\cos\theta_2}}$$

$$x = \frac{t\sin(\theta_1 - \theta_2)}{\cos\theta_2}$$

If angle of incidence θ_1 is every small, θ_2 will also be small

$\sin(\theta_1 - \theta_2) \approx (\theta_1 - \theta_2)$ and $\cos\theta_2 \approx 1$

$$x = t(\theta_1 - \theta_2)$$

$$x = t\theta_1 (1 - \frac{\theta_2}{\theta_1})$$

From Snell's law

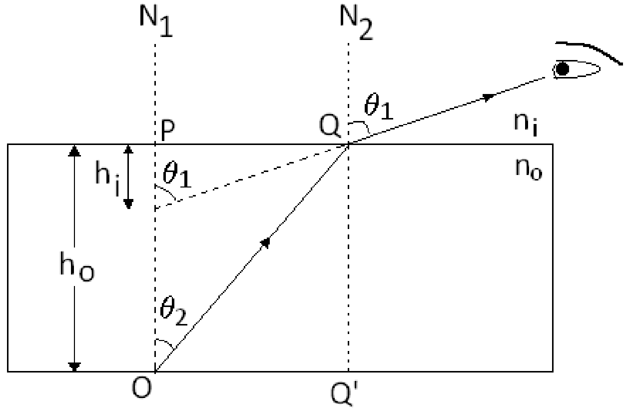
$$\frac{sin\theta_1}{sin\theta_2} = \frac{\theta_1}{\theta_2} = \frac{n_2}{n_1}$$

Thus

$$x = t\theta_1 \left(1 - \frac{n_1}{n_2}\right)$$

Real Depth and Virtual depth

Is Another example of lateral shift.



n_i is the refractive index of observer medium, n_o is the refractive index of object medium. h_o is the real depth and h_i is virtual depth.

Applying Snell's law at point Q,

$n_i sin\theta_1 = n_o sin\theta_2$For normal incidence θ_1 and θ_2 are very small.

$\therefore$ sinθ ≈θ ≈tanθ

$n_i tan\theta_1 = n_o tan\theta_2$ . .---- Eq(1) But from figure

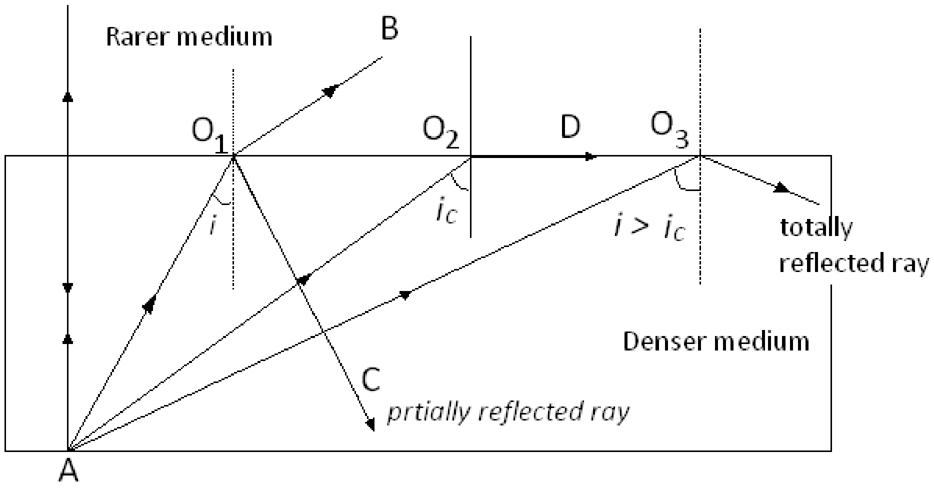
$$tan\theta_1 = \frac{PQ}{PI} = \frac{PQ}{h_i} \ and \ tan\theta_2 = \frac{PQ}{PO} = \frac{PQ}{h_o}$$

Thus from equation (1)

$$n_i \frac{PQ}{h_i} = n_o \frac{PQ}{h_o}$$

$$\frac{n_i}{n_o} = \frac{h_i}{h_o}$$

Total Internal Reflection



When light travels from optically denser medium to rarer medium at the interface, it is partly reflected back into same medium and partly refracted to the second medium. This reflection is called the *internal reflection*.
When a ray of light enters from a denser medium to a rarer medium, it bends away from the normal.
for example. As shown in figure , The incident ray AO_1 is partially reflected ($O_1 C$) and partially transmitted ($O_1 B$) or refracted.
the angle of refraction (r) being larger than the angle of incidence (i).
As the angle of incidence increases, so does the angle of refraction, till for the ray AO_2 the angle of refraction is π/2. The refracted ray is bent so much away from the normal that it grazes the surface at the interface between the two media.
This is shown by the ray $AO_2 D$ in Fig.
If the angle of incidence is increased still further (e.g, the ray AO_3), refraction is not possible. And the incident ray is totally reflected. This is called total internal reflection.

The angle of incidence for which angle of refraction is π/2 is called critical angle
From Snell's law sini / sinr = n_2 / n_1 If n_2 is air then n_1 = 1
When i = i_C (critical angle) then r = π/2. Let n_1 refractive index of denser medium = n Then
$\sin i_C = 1/n$

Solved Numerical

Q) A ray of light is incident at angle of 60^O on the face of a rectangular glass slab of the thickness 0.1m and refractive index 1.5 . Calculate the lateral shift
Solution: Here i = 60^O ; n = 1.5 and t = 0.1 m
From Snell's law

$$n = \frac{\sin i}{sinr}$$

$$sinr = \frac{sini}{n} = \frac{sin60}{1.5} = \frac{0.866}{1.5} = 0.5773$$

Thus r = $35^{O}15'$

Now lateral shift

$$x = \frac{\text{tsin}(\theta_1 - \theta_2)}{\cos\theta_2} = \frac{0.1 \times sin(60^0 - 35^0 15')}{cos35^O 15'} = 0.0512\ m$$

Q) A tank filled with water to a height of 12.5cm. The apparent depth of needle lying at the bottom of the tank is measured by a microscope to be 9.4 cm. What is the refractive index 1.63 up to the same height, by what distance would the microscope have to be moved to focus on the needle again?

Solution:

Case I : When the tank is filled with water:

Real depth h_O = 12.5 cm Apparent depth h_1 = 9.4 cm, n_i = air = 1

From formula

$$\frac{n_i}{n_o} = \frac{h_i}{h_o}$$

$$\frac{1}{n_o} = \frac{9.4}{12.5}$$

n_O = 1.33

Case II: When tank filled with the liquid of refractive index = n_O = 1.63

$$\frac{1}{1.63} = \frac{h_i}{12.5}$$

$$h_i = \frac{12.5}{1.63} = 7.67\ \ cm$$

Therefore the distance through which the microscope to be moved = 9.4 -767 = 1.73 cm

Q) A fish rising vertically to the surface of water in a lake uniformly at the rate of 3 m/s observes kingfisher bird diving vertically towards water at the rate 9 m/s vertically above it. If the refractive index of water is 4/3, find the actual velocity of the dive of the board.

Solution: Velocity of bird with respect to stationary fish in water = 6m/s

Thus apparent displacement of bird in 1 sec h_i = 6 m

n_i = 4/3, n_O = 1 (air)

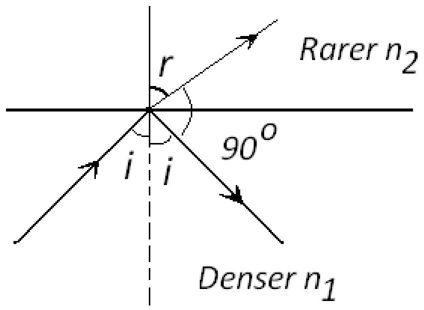
Now from formula

$$\frac{n_i}{n_o} = \frac{h_i}{h_o}$$

$$\frac{\frac{4}{3}}{1} = \frac{6}{h_o}$$

$$h_O = \frac{6 \times 3}{4} = 4.5\ m/s$$

Q) A ray of light from a denser medium strikes a rarer medium at an angle of incidence i. If the reflected and the refracted rays are mutually perpendicular to each other, what is the value of the critical angle?



Solution: From Snell's law, we have

$$\frac{\sin i}{\sin r} = \frac{n_2}{n_1} = n_{21}$$

According to given problem

i+r+90 = 190

R = 90-i

Thus from above equation

$$n_{21} = \frac{sin(90-i)}{sini} = coti$$

By definition for critical angle

$$i_c = sin^{-1}\left(\frac{1}{n_{21}}\right)$$

$$i_c = sin^{-1}\left(\frac{1}{cot\ i}\right)$$

$$i_c = sin^{-1}(\tan i)$$

Examples of total internal reflection

(i) **Mirage**: In summer, air near the surface becomes hotter than the layer above it. Thus as we go away from the surface air becomes optically denser.

Ray of light travelling from the top of tree or building towards the ground it passes through denser air layer to rarer layers of air, as a result refracted rays bend away from normal and angle of incidence on consecutive layers goes on increasing, at a particular layer angle of incidence is more than critical angle rays gets internally reflected and enters the eye of observer and observer observes inverted image of the object. Such inverted image of distant tall object causes an optical illusion to observer. This phenomenon is called mirage.

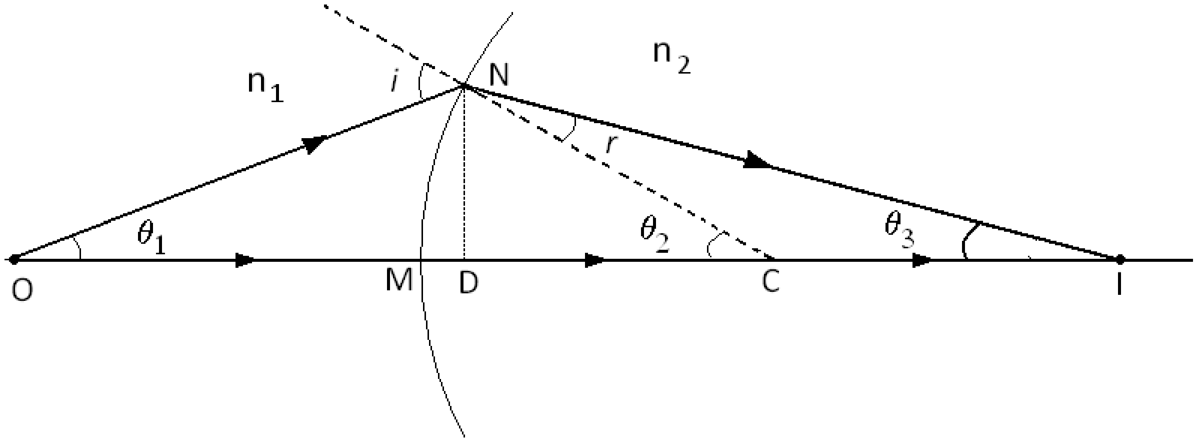
(ii) Diamond: Brilliance of diamond is due the internal reflection of light. Critical angle of diamond- air interface is (24.4^O) is very small, it is very likely to undergo total internal reflection inside it. By cutting the diamond suitably, multiple total internal reflections can be made to occur.

(iii) Prism: Prism designed to bend light by 90^O or 180^O make use of total internal reflection. Such a prism is also used to invert images without changing the size.

(iv) Optical fibres. Optical fibres make use of the phenomenon of total internal reflection. When signal in the form of light is directed at one end of the fibre at suitable angle, it undergoes repeated total internal reflections along the length of the fibre and finally comes out of the other end.

Optical fibres are fabricated with high quality composite glass/quartz fibres. Each fibre consists of a core and cladding. The refractive index of the material of the core is higher than that of the cladding. The main requirement in fabrication of optical fibres is that there should be very little absorption of light as it travels for long distances inside them. This has been achieved by purification and special preparation of materials such as as quartz. In silica fibres, it is possible to transmit more than 95 of light over a fibre length of 1km

Refraction at spherical surface and by lenses



MO is object distance = - u

MC is radius of curvature = +R

MI is image distance = +v

The ray incident from a medium of refractive index $n_{1+,}$ to another of refractive index n_2.

We take the aperture of the surface so small that we can neglect the distance MD.

$$tan\theta_1 = \frac{ND}{OD} = \frac{MN}{OM}$$

$$tan\theta_2 = \frac{ND}{CD} = \frac{MN}{CM}$$

$$tan\theta_3 = \frac{ND}{ID} = \frac{MN}{IM}$$

θ_1, θ_2, θ_3 are very small , for small θ , tan θ = θ

$$\theta_1 = \frac{ND}{OD} = \frac{MN}{OM} = \frac{MN}{-u}$$

$$\theta_2 = \frac{ND}{CD} = \frac{MN}{CM} = \frac{MN}{R}$$

$$\theta_3 = \frac{ND}{ID} = \frac{MN}{IM} = \frac{MN}{v}$$

From figure i is exterior angle i = $\theta_1 + \theta_2$ and

θ_2 is exterior angle thus $\theta_2 = r + \theta_3$ or $r = \theta_2 - \theta_3$
Applying Snell's law at point N we get
$n_1 \sin i = n_1 \sin r$
for small angles of I and r
$n_1 i = n_2 r$
$n_1 (\theta_1 + \theta_2) = n_2 (\theta_2 - \theta_3)$

$$n_1 \left(\frac{MN}{-u} + \frac{MN}{R}\right) = n_2 \left(\frac{MN}{R} - \frac{MN}{v}\right)$$

$$n_1 \left(\frac{1}{-u} + \frac{1}{R}\right) = n_2 \left(\frac{1}{R} - \frac{1}{v}\right)$$

By rearranging the terms

$$\frac{n_2}{v} - \frac{n_1}{u} = \frac{n_2 - n_1}{R}$$

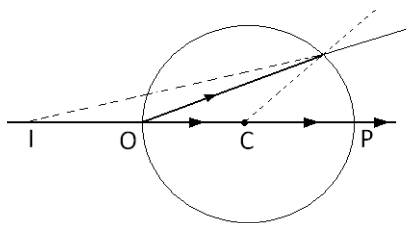
Above equation gives relation between object distance and image distance in terms of refractive index of the medium and radius of curvature of the spherical surface. It holds good for any spherical surface.

curved surface magnification

$$m = \frac{n_1}{n_2}\frac{v}{u}$$

Solved Numerical

Q) If a mark of size 0.2 cm on the surface of a glass sphere of diameter 10cm and n = 1.5 is viewed through the diametrically opposite point, where will the image be seen and of what size?



Solution:
As the mark is on one surface, refraction will occur on other surface (which is curved). Further refraction is taking place from glass to air so,
$n_2 = 1$ (air) $n_1 = 1.5$ (glass)object distance = diameter of sphere = -10cm and R = -5 cm
Using formula

$$\frac{n_2}{v} - \frac{n_1}{u} = \frac{n_2 - n_1}{R}$$

$$\frac{1}{v} - \frac{1.5}{-10} = \frac{1 - 1.5}{-5}$$

V = -20 cm

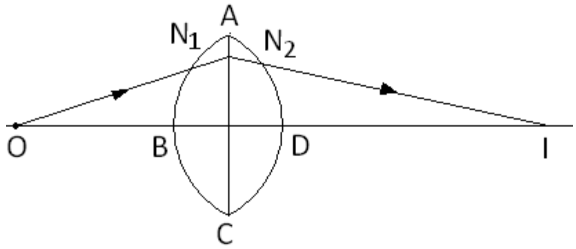
Hence image is at a distance of 20 cm from P towards Q

In case of refraction at curved surface magnification

$$m = \frac{n_1}{n_2}\frac{v}{u} = \frac{1}{1.5} \times \frac{-20}{-10} = +3$$

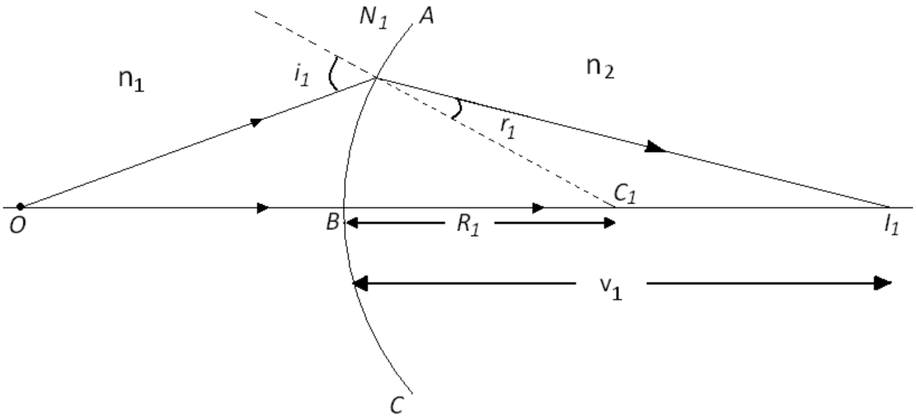
So, the image is virtual erect and of size h' = m×h = 3×0.2 = 0.6 cm

Refraction by lens



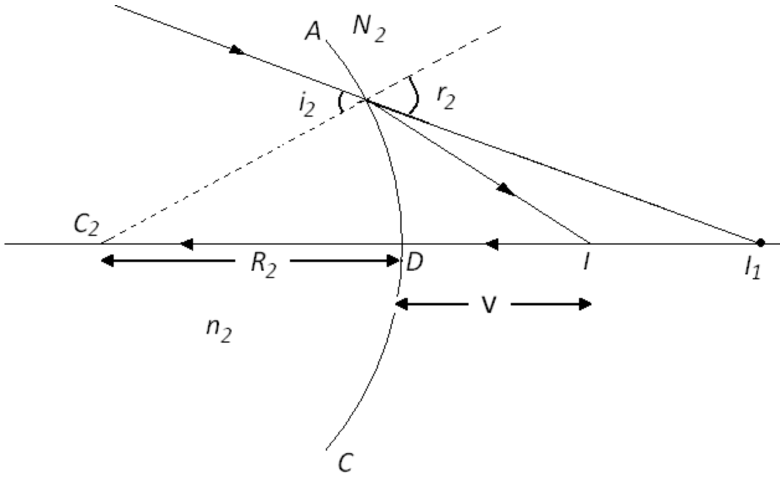
Lens shown in diagram have two curved surfaces. Surface ABC (N_1) have radius of curvature R_1 . Another surface ADC (N_2) have radius of curvature R_2. Image formed by Surface N_1 acts as an object for surface N_2 and final image is formed at I

Consider surface ABC as shown in figure here object O is in medium of refractive index n_1 and form image at I_1 at a distance v_1 in medium of refractive index n_2. from formula for refraction at curved surface.



$$\frac{n_2}{v_1} - \frac{n_1}{u} = \frac{n_2 - n_1}{R_1} \quad ..eq(1)$$

Now image I_1 which is in medium of refractive index n_2 acts like a object for surface ADC (N_2) and forms image in medium of refractive n_1 at I



here u = v_1 and radius of curvature = R_2
From formula for refraction at curved surface

$$\frac{n_1}{v} - \frac{n_2}{v_1} = \frac{n_1 - n_2}{R_2} \quad ..eq(2)$$

On adding equation (1) and (2) we get

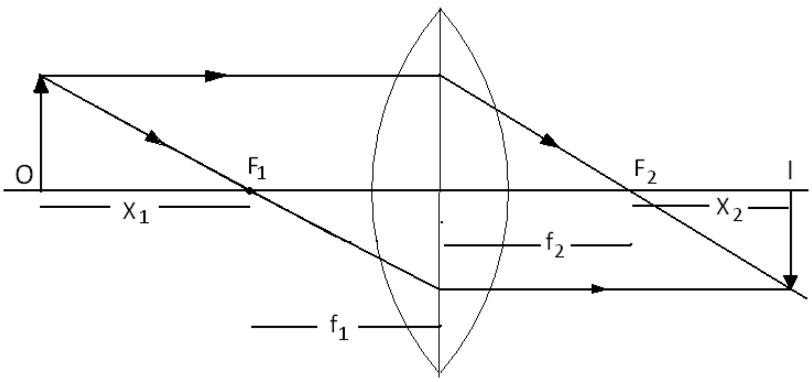
$$\frac{n_1}{v} - \frac{n_1}{u} = (n_2 - n_1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \quad ..eq(3)$$

$$\frac{1}{v} - \frac{1}{u} = \left(\frac{n_2}{n_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \quad ..eq(4)$$

Here $\frac{n_2}{n_1}$ is refractive index of lens material with respect to surrounding.
Lens-Makers's formula
If u = ∞ the image will be formed at f thus from equation 4

$$\frac{1}{f} = \left(\frac{n_2}{n_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \quad ..eq(5)$$

Magnification
m = v/u
Newton's formula:



x_1 and x_2 are known as extra focal distance and extra focal image distance.

$x_1 . x_2 = f_1 f_2$

if $f_1 = f_2 = f$ then

$x_1 . x_2 = f^2$

Power of lens: If defined as the tangent of the angle by which it converges or diverges a beam of light falling at unit distant from the optical centre

$$tan\delta = \frac{h}{f}$$

If h = 1 then

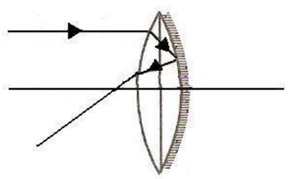
$$tan\delta = \frac{1}{f}$$

For small value of δ

$$\delta = \frac{1}{f}$$

Thus P = 1 / f SI unit for power of lens is diopter (D) 1D = 1 m^{-1}

Lens with one surface silvered



When one surface is silvered, the rays are reflected back at this silvered surface and set up acts as a spherical mirror

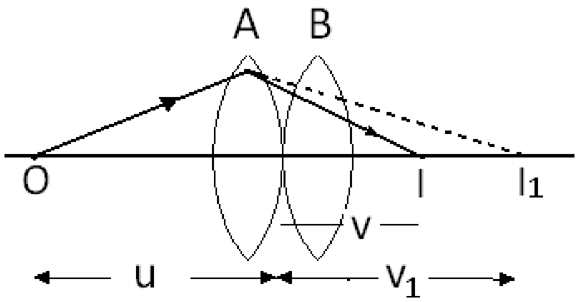
Focal length is given by formula

$$\frac{1}{f} = \frac{1}{f_l} + \frac{1}{f_m} + \frac{1}{f_l}$$

Here f_l is focal length of lens , f_m is focal length of mirror

In the above formula, **the focal length of converging lens or mirror is taken positive and that of diverging lens or mirror is taken as negative.**

Combination of lenses in contact



Consider two lenses A and B of focal length f_1 and f_2 placed in contact with each other. Let the object be placed at a point O beyond the focus of the first lens A.

The first lens produces the image at I_1. Since the image is real. It serves as object for second lens B produces image at I

The direction of rays emerging from the first lens gets modified in accordance with the angle at which they strike the second lens.

We assume the optical centres of the thin lens coincident.

For image formed by lens A. we get

$$\frac{1}{v_1} - \frac{1}{u} = \frac{1}{f_1}$$

For the image formed by second lens B, we get

$$\frac{1}{v} - \frac{1}{v_1} = \frac{1}{f_2}$$

Adding above equations

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f_1} + \frac{1}{f_2}$$

If two lens-system is regarded as equivalent to a single lens of focal length f, we have

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

So we have

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}$$

The derivation is valid for any number of thin lenses in contact

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3} \ldots \ldots \ldots$$

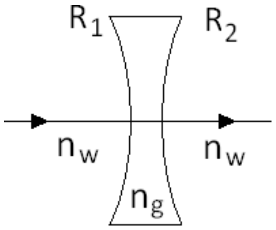
In terms of equivalent power

$P = P_1 + P_2 + P_3 + \ldots\ldots..$

Image form by first lens is object for second lens it implies that total magnification for combination of lenses is

$m = m_1 m_2 m_3$ ..... (here m_1 and m_2...etc are magnification of individual lenses)

Solved Numerical



Q) Calculate the focal length of a concave lens in water (Refractive index = 4/3) if the surface have radii equal to 40cm and 30 cm (refractive index of glass = 1.5)

Solution: R_1 = -30cm R_2 = +40cm

We have

$$\frac{1}{f} = \left(\frac{n_2}{n_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\frac{1}{f} = \left(\frac{1.5}{4/3} - 1\right)\left(\frac{1}{-30} - \frac{1}{40}\right)$$

$$\frac{1}{f} = -\frac{960}{7} = -131.7\ cm$$

Q) A plano-covex lens has focal length 12cm and is made up of glass with refractive index 1.5. Find the radius of curvature of its curved side

Solution: Let R_1 be the radius of curvature, $R_2 = \infty$, f = +12cm

From formula for focal length

$$\frac{1}{f} = (\frac{n_2}{n_1} - 1)\ (\frac{1}{R_1} - \frac{1}{R_2})$$
$$\frac{1}{12} = (\frac{1.5}{1} - 1)\ (\frac{1}{R_1} - \frac{1}{\infty})$$

R_1 = 6 cm

Q) A magnifying lens has a focal length of 10cm (i) Where should the object be placed if the image is to be 30cm from the lens? (ii) What will be the magnification

Solution: For a convergent lens , If image formed on object side ,

Thus v = -30 cm f = +10cm

Let x be the distance of object

Using lens formula

$$\frac{1}{f} = \frac{1}{v} - \frac{1}{u}$$
$$\frac{1}{10} = \frac{1}{-30} - \frac{1}{-x}$$

x= 7.5 cm

(ii) magnification m = v/u = -30/-7.5 = +4

Thus image is virtual erect and four times the size of the object.

Q) An object 25 cm high is placed in front of a convex lens of focal length 30 cm. If the height of the image formed is 50 cm, find the distance between the object and the image

Solution

We have

|m| = h'/h = v/u = 50/25 = 2

There are two possibilities

(i) if the image is inverted (i.e.) real

M = V/u = -2 or V = -2u

Let x be the object distance, in this case

We have u = -x, v = +2x f = + 30cm

Using lens formula

$$\frac{1}{f} = \frac{1}{v} - \frac{1}{u}$$
$$\frac{1}{30} = \frac{1}{2x} - \frac{1}{-x}$$

x = 45 cm then v = 90 cm

Hence distance between object and image is = 45 + 90 = 135

(ii) If the image is erect (i.e virtual)

m = v/u = +2

Let x' be the object distance , we have

U = -x'; v = -2x' ; f = 30 cm

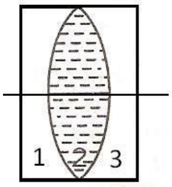
Using the lens formula, we have

$$\frac{1}{30} = \frac{1}{-2x'} - \frac{1}{-x'}$$

x' = 15 cm

Hence distance between object and image = 15 cm

Q) Two plano-concave lenses made of glass of refractive index 1.5 have radii of curvature 20 cm and 30cm. they are placed in contact with curved surface towards each other and the space between them is filled with liquid of refractive index 4/3. Find the focal length of the system



Solution: As shown in figure, the system is equivalent to combination of three lenses in contact

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3}$$

By lens maker's formula

$$\frac{1}{f} = (\frac{n_2}{n_1} - 1)\ (\frac{1}{R_1} - \frac{1}{R_2})$$

$$\frac{1}{f_1} = (\frac{3}{2} - 1)\ (\frac{1}{\infty} - \frac{1}{20}) = \frac{1}{40} cm$$

$$\frac{1}{f_2} = (\frac{4}{3} - 1)\ (\frac{1}{20} - \frac{1}{-30}) = \frac{5}{180} cm$$

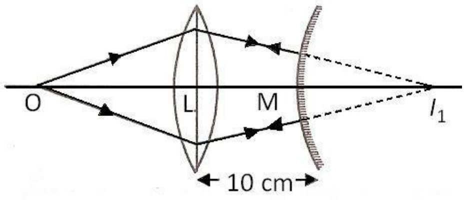
$$\frac{1}{f_3} = (\frac{3}{2} - 1)\ (\frac{1}{-30} - \frac{1}{\infty}) = \frac{-1}{60} cm$$

Now

$$\frac{1}{F} = \frac{1}{40} + \frac{1}{180} + \frac{-1}{60} = \frac{-1}{72}$$

F = -72 cm

Q) A point object is placed at a distance of 12 cm on the axis of convex lens of focal length 10cm. On the other side of the lens, a convex mirror is placed at the distance of 10 cm from the lens such that the image formed by the combination coincides with the object itself. What is the focal length of the mirror.?



Solution :
For the refraction at the convex lens, we have u= -12 cm, v = ? f = + 10 cm
Using lens formula, we have

$$\frac{1}{f} = \frac{1}{v} - \frac{1}{u}$$

$$\frac{1}{10} = \frac{1}{v} - \frac{1}{-12}$$

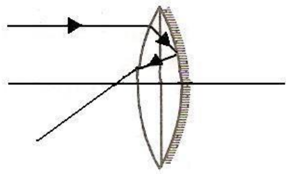
V = +60cm

Thus, in the absence of convex mirror, convex lens will form the image at I_1 , at a distance 60 cm behind the mirror. This image will act as object for mirror t, object distance for mirror = 60-10 = +50cm

Now as the final image I is formed at the object itself, indicates that the rays on the mirror are incident normally i.e image I_1 is at the centre of curvature of mirror

So R = 50 cm and f = R/2 = 50/2 = +25cm

Q) One face of an equiconvex lens of focal length 60cm made of glass (Refractive index 1.5) is silvered. Does it behave like a concave mirror or convex mirror



Solution : Let x be the radius of curvature of each surface from lens maker's formula

$$\frac{1}{f} = (\frac{n_2}{n_1} - 1) \ (\frac{1}{R_1} - \frac{1}{R_2})$$

$$\frac{1}{60} = (1.5 - 1) \ (\frac{1}{x} - \frac{1}{-x})$$

$$\frac{1}{60} = 0.5 \times \frac{2}{x}$$

X = 60 cm

Let f be the focal length of the equivalent spherical mirror. Then

$$\frac{1}{f} = \frac{1}{f_l} + \frac{1}{f_m} + \frac{1}{f_l}$$

Since lens is equiconvex

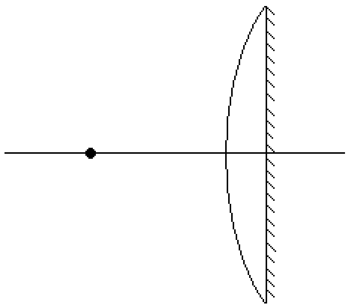
$$\frac{1}{f} = \frac{2}{f_l} + \frac{1}{f_m}$$

Here f_l = +60 cm (convex lens) , f_m =R/2 = +30 (concave mirror)

$$\frac{1}{f} = \frac{2}{60} + \frac{1}{30} = \frac{1}{15}$$

F = +15cm cm

The positive sign indicates that the resulting mirror is concave



Q) The plane surface of a plano-convex lens of focal length 60 cm is silvered . A point object is placed at a distance 20cm from the lens. Find the position and nature of the final image formed

Solution:

Let f be the focal length of the equivalent spherical mirror

We have

$$\frac{1}{f} = \frac{1}{f_l} + \frac{1}{f_m} + \frac{1}{f_l}$$

Or

$$\frac{1}{f} = \frac{2}{f_l} + \frac{1}{f_m}$$

Here f_l = + 60 cm, $f_m = \infty$

$$\frac{1}{f} = \frac{2}{60} + \frac{1}{\infty} = \frac{1}{30}$$

Or f = + 30 cm

The problem is reduced to a simple case where a point object is placed in front of a cocave mirror/

Now, using the mirror formula

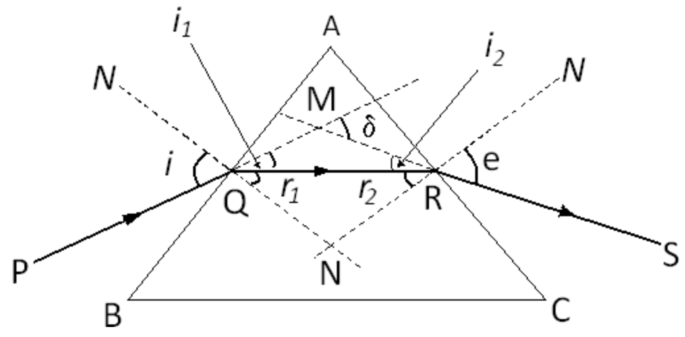
$$\frac{1}{f} = \frac{1}{v} + \frac{1}{u}$$

$$\frac{1}{-30} = \frac{1}{v} + \frac{1}{-20}$$

V = 60 cm

SECTION III
REFRACTION THROUGH A PRISM

REFRACTION THROUGH A PRISM



PQ is incident ray and angle made by PQ with normal is i. QR is refracted ray at surface BA and angle of refraction is r_1

For surface AC ray QR is incident ray and angle of incidence is r_2 .. RS refracted ray and angle is called angle of emergence

The angle between the emergent ray and RS and the direction of incident ray PQ is called the angle of deviation δ.

In quadrilateral AQNR, two angles (at vertices Q and R) are right angles. Therefore the sum of other angles of quadrilateral is 180^O

$\angle A + \angle QNR = 180^O$ --eq(1)

From the triangle QNR

$r_1 + r_2 + \angle QNR = 180^O$ --eq(2)

from equation (1) and (2)

$r_1 + r_2 = A$ --eq(3)

Now $\delta = i_1 + i_2$ (as δ is exterior angle) –eq(4)

Now $i_1 = (i - r_1)$ and $i_2 = (e - r_1)$ (from geometry of figure)

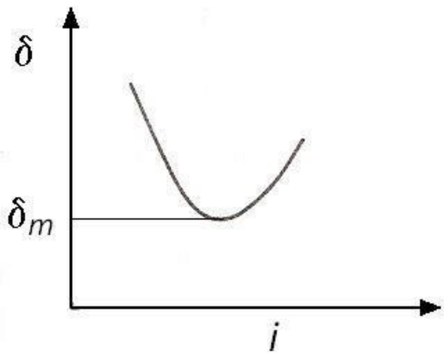
From equation 3

$\delta = (i - r_1) + (e - r_1)$

$\delta = (i + e) - (r_1 + r_1)$

$\delta = (i + e) - A$ --eq(5)

Thus , the angle of incidence depends on the angle of incidence.



From the graph δ vs i we can find that for every angle of deviation there are two values of angle of incidence. It means that vale of i and e are interchangeable. Except i=e . At i =e the angle of deviation is minimum denoted by δ_m or D_m, at angle of minimum deviation ray inside the prism becomes parallel.

Calculation of refractive index of Prism

By applying Snells law at the surface BA we get

$n_1 \sin i = n_2 \sin r_1$

$$n_{21} = \frac{n_2}{n_1} = \frac{sini}{sinr}$$

At angle of minimum deviation I = e implies $r_1 = r_2 = r$ From equation (3)

2r = A or r = A/2 ---eq(6)
From equation (5)
D_m= 2i – A or i = ($A+D_m$)/2 –eq(7)

Substituting values from equation 6 and 7

$$n_{21} = \frac{\sin\frac{A+D_m}{2}}{\sin\frac{A}{2}}$$

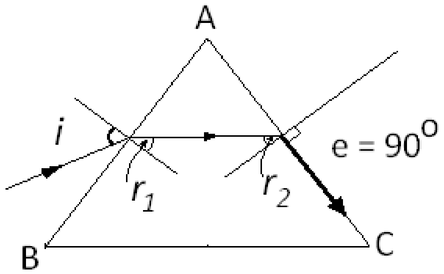
For small angle prism D_m is also very small and we get

$$n_{21} = \frac{\sin\frac{A+D_m}{2}}{\sin\frac{A}{2}} = \frac{\frac{A+D_m}{2}}{\frac{A}{2}} = \frac{A+D_m}{A}$$

$D_m = (n_{21} - 1) A$
It implies that, thin prism do not deviate light much

Solved Numerical

Q) A ray of light falls on one side of a prism whose refracting angle is 60^O . Find the angle of incidence in order that the emergent ray just graze the other side (Refractive index = 3/2)



Solution: Given A = 60^O, e = 90^O
∴ r_2 = C the critical angle of the prism
Now n = 1 /sinC
Or sinC = 1/n = 2/3
C = 41^{O}49'
Again, A = $r_1 + r_2$
r_1 = A - r_2 = 60^O – 41^{O}49' = 18^O 11'
For the refraction at surface AB we have

$$n = \frac{\sin i}{\sin r_1}$$

sin i = n sinr_1
sin i = 1.5 × sin 18^O 11'
sin i = 1.5 × 0.312 = 0.468
i = 27^{O}55'
Q) The refractive index of the material of a prism of refracting angle 45^O is 1.6 for a certain monochromatic ray. What should be the minimum angle of incidence of this ray on the prism so that no that no total internal reflection takes place as they come out of the prism
Solution:
Given A = 45^O , n = 1.6
We have
Sin C = 1/n
sinC = 1/1.6

$C = 38.68^O$

For total internal reflection not take place at the surface AC, we have

$r_2 \leq C$

or $(r_2)_{max} = C$

Now $r_1 + r_2 = A$

Or $r_1 = (A - r_2)$

Or $(r_1)_{mini} = A - (r_2)_{max} = 45^O - 38..68^O = 6.32^O$

For the refraction at the first face

We have

$$n = \frac{sini_1}{sinr_1}$$

Or $sini_1 = n\ sinr_1$

Sin $i_1 = 1.6 \times \sin(6.32^O) = 0.176$

$i_1 = 10.14^O$

Q) Find the minimum and maximum angle of deviation for a prism with angle A = 60^O and n = 1.5

Solution : Minimum deviation:

The angle of minimum deviation occurs when i = e and $r_1 = r_2$ and is given by

$$n = \frac{sin\frac{(A + \delta_m)}{2}}{sin\frac{A}{2}}$$

$$\delta_m = 2sin^{-1}(nsin\frac{A}{2}) - A$$

Substituting n = 1.5 and A = 60^O, we get

$\delta_m = 2sin^{-1}(0.75) - 60 = 37^O$

Maximum deviation

The deviation is maximum when i = 90^O or e = 90^O

Let i = 90^O

$r_1 = C = sin^{-1}(1/n)$

$r_1 = sin^{-1}(2/3) = 42^O$

$r_2 = A - r_1 = 60^O - 42^O = 18^O$

Using

$$\frac{sinr_2}{sine} = \frac{1}{n}$$

Sine = $nsinr_2$

e = 28^O

∴ Deviation = $\delta_{max} = (I + e) - A = 90 + 2\ 8 - 60 = 58^O$

Dispersion by a prism

The phenomenon of splitting of light into its component colours is known as dispersion.

Different colours have different wavelengths and different wavelengths have different velocity in transparent medium.
Longer wave length have more velocity compared to shorter wavelength. Thus refractive index is more for shorter vale length than longer wavelength.
Red colour light have highest wave length in visible light spectrum and hence lowest refractive index.
Violet colour light have smallest wave length in visible light spectrum., it have maximum refractive index.
Thick lens could be considered as maid of many prism, therefore, thick lens shows chromatic aberration due to dispersion of light

Some natural phenomena due to sunlight

THE RAINBOW

Primary rainbow

Rainbow is formed due to two times refraction and one time total internal reflection in water drop present in atmosphere
Sunlight is first refracted as it enters a raindrop, which causes the different wavelengths of white light to separate. Longer wavelength of light (red) are bent the least while the shorter wavelength (violet) are bent the most. Next, these component rays strike the inner surface of the water droplet and gets internally reflected if the angle of incidence is greater than the critical angle. The reflected light is again refracted as it comes out of the drop.
It is found that violet light emerges at an angle 40^O related to the incoming sunlight and red light emerges at an angle of 42^O . For other colours , angle lies between these two values.

Secondary rainbow

When light rays undergoes two internal refraction inside the rain drop, a secondary raindrop is formed. Its intensity is less compared to primary due to four stem process. Violet ray makes an angle of 53^O and red light rays makes 50^O with related to incoming sunlight. The order of colours is reversed

Rayleigh scattering

The amount of scattering is inversely proportional to the fourth power of the wavelength. This is known as **Rayleigh scattering**

Why sky is blue

As sunlight travels through the earth's atmosphere, it gets scattered (changes its direction) by the atmospheric particles. Light of shorter wavelengths is scattered much more than light of longer wavelengths. The amount of scattering is inversely proportional to the fourth power of the wavelength. This is known as **Rayleigh scattering**. Hence, the bluish colour predominates in a clear sky, since blue has a shorter wavelength than red and is scattered much more strongly. In fact, violet gets scattered even more than blue, having a shorter wavelength. But since our eyes are more sensitive to blue than violet, we see the sky blue.

Why clouds look white
Large particles like dust and water droplets present in the atmosphere behave differently. The relevant quantity here is the relative size of the wavelength of light λ, and do not follow **Rayleigh scattering** . For large scattering objects(for example, raindrops, large dust or ice particles) all wavelengths are scattered nearly equally. Thus, clouds which have droplets of water with larger particle are generally white.

At sunset or sunrise, Sun looks reddish
Sun rays have to pass through a larger distance in the atmosphere (Fig. 9.28).Most of the blue and other shorter wavelengths are removed by scattering. The least scattered light reaching our eyes, therefore, the sun looks reddish. This explains the reddish appearance of the sun and full moon near the horizon.

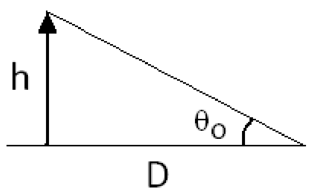
SECTION III
OPTICAL INSTRUMENTS

OPTICAL INSTRUMENTS
Eye
Function of ciliary muscles: The shape (curvature) and therefore the focal length of the lens can be modified somewhat by the ciliary muscles
For example, when the muscle is relaxed, the focal length is about 2.5 cm and objects at infinity are in sharp focus on the retina. When the object is brought closer to the eye, in order to maintain the same image-lens distance (≅ 2.5 cm), the focal length of the eye lens becomes shorter by the action of the ciliary muscles.
Accommodation: Property of eye to change the focal length as required is called accommodation.
Retina: The retina is a film of nerve fibers covering the curved back surface of the eye. The retina contains rods and cones which sense light intensity and colour, respectively, and transmit electrical signals via the optic nerve to the brain which finally processes this information.. For example, when the muscle is relaxed, the focal length is about 2.5 cm and objects at infinity are in sharp focus on the retina. When the object is brought closer to the eye, in order to maintain the same image-lens distance (≅ 2.5 cm), the focal length of the eye lens becomes shorter by the action of the ciliary muscles.
Least distance of distinct vision: If the object is too close to the eye, the lens cannot curve enough to focus the image on to the retina, and the image is blurred. The closest distance for which the lens can focus light on the retina is called the least distance of distinct vision, or the near point. The standard value for normal vision is taken as 25 cm. (Often the near point is given the symbol D.) This distance increases with age, because of the decreasing effectiveness of the ciliary muscle and the loss of flexibility of the lens. The near point may

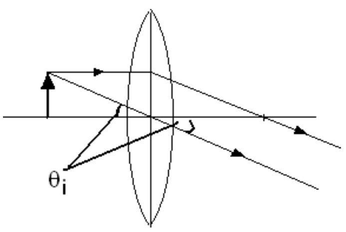
be as close as about 7 to 8 cm in a child ten years of age, and may increase to as much as 200 cm at 60 years of age.
Presbyopia: If an elderly person tries to read a book at about 25 cm from the eye, the image appears blurred. This condition (defect of the eye) is called presbyopia. It is corrected by using a converging lens for reading.
Myopia: the light from a distant object arriving at the eye-lens may get converged at a point in front of the retina. This type of defect is called nearsightedness or myopia. This means that the eye is producing too much convergence in the incident beam. To compensate this, we interpose a concave lens between the eye and the object, with the diverging effect desired to get the image focused on the retina

Simple Microscope

A simple magnifier or microscope is a converging lens of small focal length.
The least distance at which a small object can be seen clearly with comfort is known as near point (D) or distance of most distinct vision. For normal eye this distance is 25 cm.
Suppose a linear object with height h_O is kept at near point (i.e. u = D = 25 cm) from eye. Let it subtend an angle θ_O with eye .
The magnification when the image is at infinity.
Now, if object is kept is kept at the focal length (f) of a convex lens such that its virtual image is formed at a infinity. In this case we will have to obtained the *angular* magnification.
Suppose the object has a height *h*. The maximum angle it can subtend, and be clearly visible (without a lens), is when it is at the near point, i.e., a distance *D.* The angle subtended is then given by



$$tan\theta_O \approx \theta_O = \frac{h}{D}$$



We now find the angle subtended at the eye by the image when the object is at *u*. From the relations $m = \frac{h'}{h} = \frac{v}{u}$

$$h' = \frac{v}{u}h$$

we have the angle subtended by the image

$$tan\theta_i \approx \theta_i = \frac{h'}{-v} = \frac{v}{u}h\left(\frac{1}{-v}\right) = \frac{h}{-u}$$

The angle subtended by the object, when it is at *u* = –*f*.

$$\theta_i = \frac{h}{f}$$

The angular magnification is, therefore

$$m = \frac{\theta_i}{\theta_O} = \frac{D}{f}$$

The magnification when the image is at the closest comfortable distance

If the object is at a distance slightly less than the focal length of the lens, the image is virtual and closer than infinity.

Although the closest comfortable distance for viewing the image is when it is at the near point (distance D ≅ 25 cm), it causes some strain on the eye. Therefore, the image formed at infinity is often considered most suitable for viewing by the relaxed eye. The linear magnification m, for the image formed at the near point D, by a simple microscope can be obtained by using the relation.

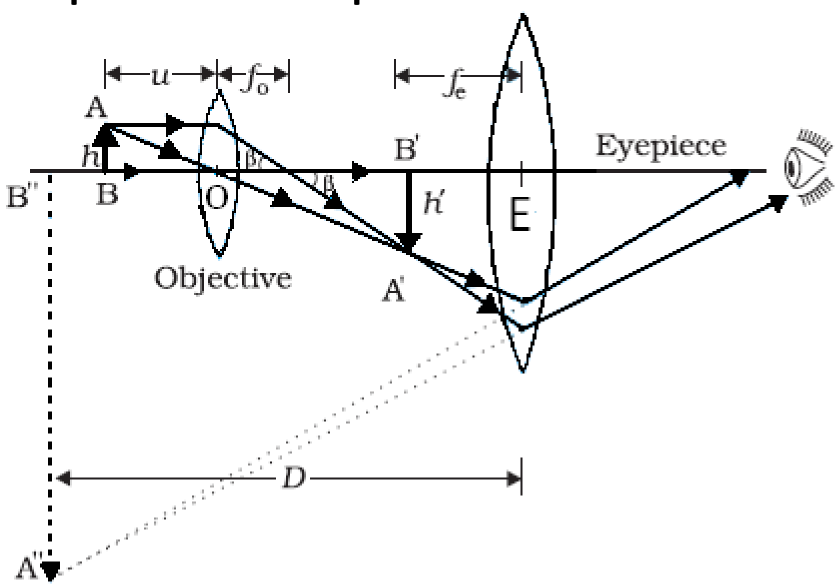
$$m = \frac{v}{u} = v\,(\frac{1}{v} - \frac{1}{f}) = (1 - \frac{v}{f})$$

Now according to our sign convention, *v* is negative, and is equal in magnitude to *D.* Thus, the magnification is

$$m = (1 + \frac{D}{f})$$

A simple microscope has a limited maximum magnification (≤ 9) for realistic focal lengths

Compound microscope



A simple microscope has a limited maximum magnification (≤ 9) for realistic focal lengths. For much larger magnifications, one uses two lenses, one compounding the effect of the other. This is known as a *compound microscope*.

A schematic diagram of a compound microscope is shown in Fig..The lens nearest the object, called the *objective*, And lens near to eye is called *eye piece*

Objective forms a real, inverted, magnified image of the object. A'B'.

This serves as the object for the *eyepiece*, which functions essentially like a simple microscope or magnifier, produces the final image, which is enlarged and virtual A"B"

the distance between the second focal point of the objective and the first focal point of the eyepiece (focal length f_e) is called the **tube length** (L) of the compound microscope

magnification

The ray diagram of Fig. shows that the (linear) magnification due to the objective,

$$tan\beta = (\frac{h}{f_o}) = (\frac{h'}{L})$$

$$m_o = \frac{h'}{h} = \frac{L}{f_o}$$

Here h' is the size of the first image, the object size being h and f_o being the focal length of the objective.

The first image is formed near the focal point of the eyepiece. Magnification due to eye piece which behaves as simple microscope is given by

$$m_e = (1 + \frac{D}{f_e})$$

When the final image is formed at infinity, the angular magnification due to the eyepiece

$$m_e = (\frac{D}{f_e})$$

Thus, the total magnification, when the image is formed at infinity, is

$$m = m_o m_e = (\frac{L}{f_o})\,(\frac{D}{f_e})$$

Thus, the total magnification, when the image is formed at near point, is

$$m = m_o m_e = (\frac{L}{f_o})\,(1 + \frac{D}{f_e})$$

Telescope

The telescope is used to provide angular magnification of distant objects.

Lens towards object is objective have larger focal length than eye piece.

Light from a distant object enters the objective and a real image is formed in the tube at its second focal point. The eyepiece magnifies this image producing a final inverted image.

The magnifying power *m* is the ratio of the angle β subtended at the eye by the final image to the angle α which the object subtends at the lens or the eye. Hence

$$m \approx \frac{\beta}{\alpha} = \frac{h}{f_e}\frac{f_o}{h} = \frac{f_o}{f_e}$$

In this case, the length of the telescope tube is *fo + fe .*

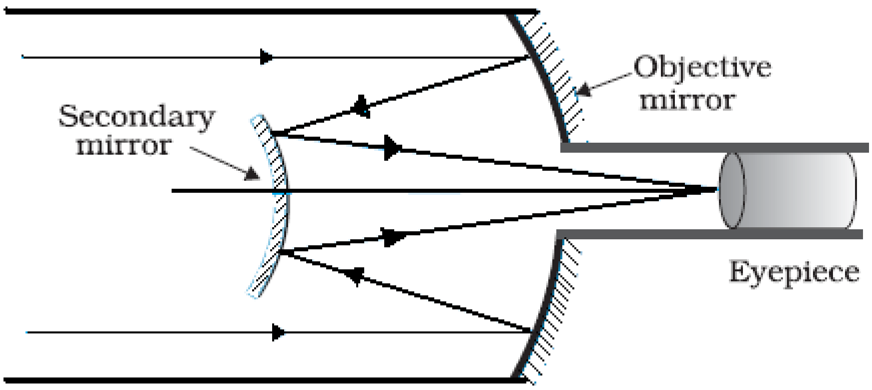
Terrestrial telescopes have, in addition, a pair of inverting lenses to make the final image erect. Refracting telescopes can be used both for terrestrial and astronomical observations.

If the final image is formed at the least distance of distinct vision, then magnifying power is

$$m = \frac{f_o}{f_e}(1 + \frac{f_e}{D})$$

The main considerations with an astronomical telescope are its light gathering power and its resolution or resolving power. The former clearly depends on the area of the objective. With larger diameters, fainter objects can be observed.
The **resolving power**, or the ability to observe two objects distinctly, which are in very nearly the same direction, also depends on the diameter of the objective. So, the desirable aim in optical telescopes

Reflecting telescope



Telescopes with mirror objectives are called *reflecting* telescopes.
They have several advantages over refracting telescope
First, there is no chromatic aberration in a mirror.
Second, if a parabolic reflecting surface is chosen, spherical aberration is also removed.
Mechanical support is much less of a problem since a mirror weighs much less than a lens of equivalent optical quality, and can be supported over its entire back surface, not just over its rim.
One obvious problem with a reflecting telescope is that the objective mirror focuses light inside the telescope tube. One must have an eyepiece and the observer right there, obstructing some light (depending on the size of the observer cage). This is what is done in the very large 200 inch (~5.08 m) diameters, Mt. Palomar telescope, California.
The viewer sits near the focal point of the mirror, in a small cage. Another solution to the problem is to deflect the light being focused by another mirror. One such arrangement using a convex secondary mirror to focus the incident light, which now passes through a hole in the objective primary mirror, is shown

Solved Numerical

Q) A compound microscope with an objective of 2.0 cm focal length and an eye piece of 4.0 cm focal length, has a tube length of 40cm. Calculate the magnifying power of the microscope, if the final image is formed at the near point of the eye

Solution Formula for magnification when image is formed at near point of eye

$$m = m_o m_e = \left(\frac{L}{f_o}\right)\left(1+\frac{D}{f_e}\right)$$

$$m = \left(\frac{40}{2}\right)\left(1+\frac{25}{4}\right) = 145$$

Q) A compound microscope consists of an objective of focal length 1cm and eyepiece of focal length 5cm separated by 12.2 cm (a) At what distance from the objective should an object should be placed to focus it properly so that the final image is formed at the least distance of clear vision (25cm)? (b) Calculate the angular magnification in this case.

Solution : From lens formula

$$\frac{1}{f} = \frac{1}{v} + \frac{1}{u}$$

Here f = 1 cm ,; u= u_e , v = - 25cm

$$\frac{1}{5} = \frac{1}{25} + \frac{1}{u_e}$$

U_e = -4.2 cm

V_O = L - | u_e| = (12.2-4.2) cm = 8 cm

From lens formula

$$\frac{1}{1} = \frac{1}{8} + \frac{1}{u_O}$$

U_O = -1.1 cm

From formula for angular magnification =

$$m = \left(\frac{v_O}{u_O}\right)\left(1+\frac{D}{f_e}\right)$$

$$m = \left(\frac{8}{-1.1}\right)\left(1+\frac{25}{5}\right) = -43.6$$

Q) The separation between the objective (f = 0.5cm) and the eyepiece (f = 5cm) of a compound microscope is 7cm. Where should a small object be placed so that the eye is least strained to see the image? Find the angular magnification produced by the microscope

Solution: The eye will strain least if the final image is formed at infinity> In this case the image formed by the objective shall fall at the focus of the eye piece.

Now v_O = L - v_e =(7-5) cm = 2cm

From lends formula for objective lens

$$\frac{1}{f} = \frac{1}{v} + \frac{1}{u}$$

$$\frac{1}{0.5} = \frac{1}{2} + \frac{1}{u_o}$$

u_O = -2/3 cm

angular magnification for image at infinity

$$m = (\frac{v_O}{u_O})\,(\frac{D}{f_e})$$

$$m = (\frac{2 \times 3}{-2})\,(\frac{25}{5}) = -15$$

Q) An astronomical telescope, in normal adjustment position, has magnifying power 5. The distance between the objective and the eyepiece is 120cm. Calculate the focal lengths of the objective and of the eyepiece.

Solution: m = f_O / f_e

f_O = m × f_e

f_O = 5 f_e

f_O + f_e = 120

thus f_e = 20 cm and f_e = 100 cm

Q) The focal length of the objective of an astronomical telescope is 75 cm and that of the eye piece is 5 cm. If the final image is formed at the least distance of distinct vision from eye, calculate the magnifying power of telescope

Solution:

$$m = \frac{f_o}{f_e}(1 + \frac{f_e}{D})$$

$$m = \frac{75}{5}(1 + \frac{5}{25}) = 18$$

SEMICONDUCTOR ELECTRONICS:

MATERIALS, DEVICES AND SIMPLE CIRCUITS

Semiconductors

It has been observed that certain materials like germanium, silicon etc. have resistivity between good conductors like copper and insulators like glass. These materials are known as semiconductors. A material which has resistivity between conductors and insulators is known as semiconductor. The resistivity of a semiconductor lie approximately between 10^{-2} and 10^4 Ω m at room temperature. The resistance of a semiconductor decreases with increase in temperature over a particular temperature range. This behavior is contrary to that of a metallic conductor for which the resistance increases with increase in temperature.

The elements that are classified as semiconductors are Si, Ge, In, etc. Germanium and silicon are most widely used as semiconductors

Energy band in solids

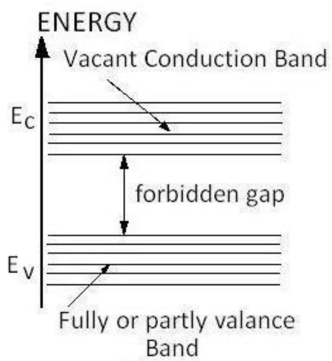
In the case of a single isolated atom, there are various discrete energy levels. In solids, the atoms are arranged in a systematic space lattice and each atom is influenced by neighboring atoms. The closeness of atoms results in the intermixing of electrons of neighboring atoms.

Inside the crystal each electron has a unique position and no two electrons see exactly the same pattern of surrounding charges. Because of this, each electron will have a different *energy level*.

These different energy levels with continuous energy variation form what are called *energy bands*. The energy band which includes the energy levels of the valence electrons is called the *valence band*. The energy band above the valence band is called the *conduction band*.

Energy difference between energy of conduction band and valance band is called band gap energy or forbidden energy gap

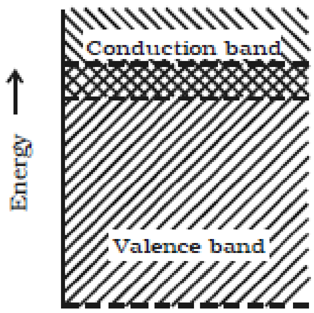
With no external energy, all the valence electrons will reside in the valence band. If the lowest level in the conduction band happens to be lower than the highest level of the valence band, the electrons from the valence band can move into the conduction band



Let us consider what happens in the case of Si or Ge crystal containing *N* atoms. For Si, the outermost orbit is the third orbit (*n* = 3), while for Ge it is the fourth orbit (*n* = 4). The number of electrons in the outermost orbit is 4 (3*s* and 3*p* electrons for Si). Hence, the total number of outer electrons in the crystal is 4*N*. The maximum possible number of electrons in the outer orbit is 8 (2*s* + 6*p* electrons).

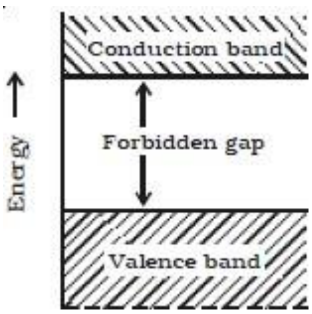
So, for the 4*N* valence electrons there are 8*N* available energy states. These 8*N* discrete energy levels can either form a continuous band or they may be grouped in different bands depending upon the distance between the atoms in the crystal

Conductors:



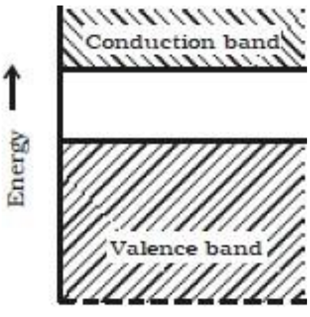
Normally the conduction band is empty. But when it overlaps on the valence band electrons can move freely into it. This is the case with metallic conductors.

Insulators



In an insulator, the forbidden energy gap is very large. In general, the forbidden energy gap is more than 3eV and almost no electrons are available for conduction. Therefore, a very large amount of energy must be supplied to a valence electron to enable it to move to the conduction band. In the case of materials like glass, the valence band is completely filled at 0 K. The energy gap between valence band and conduction band is of the order of 10 eV. Even in the presence of high electric field, the electrons cannot move from valence band to conduction band. If the electron is supplied with high energy, it can jump across the forbidden gap. When the temperature is increased, some electrons will move to the conduction band. This is the reason, why certain materials, which are insulators at room temperature become conductors at high temperature. The resistivity of insulator approximately lies between 10^{11} and 10^{16} Ω m

Semiconductors



In semiconductors (Fig), the forbidden gap is very small. Germanium and silicon are the best examples of semiconductors. The forbidden gap energy is of the order of 0.7eV for Ge and 1.1eV for Si.

There are no electrons in the conduction band. The valence band is completely filled at 0 K. With a small amount of energy that is supplied, the electrons can easily jump from the valence band to the conduction band.

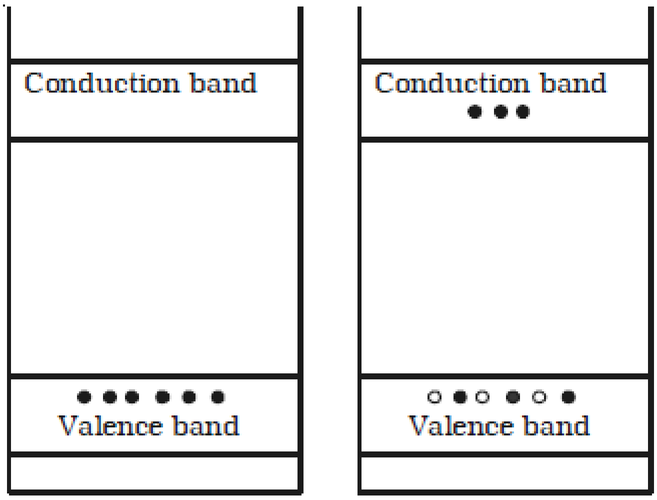
For example, if the temperature is raised, the forbidden gap is decreased and some electrons are liberated into the conduction band.

The conductivity of a semiconductor is of the order of 10^2 mho m^{-1}

INTRINSIC SEMICONDUCTOR

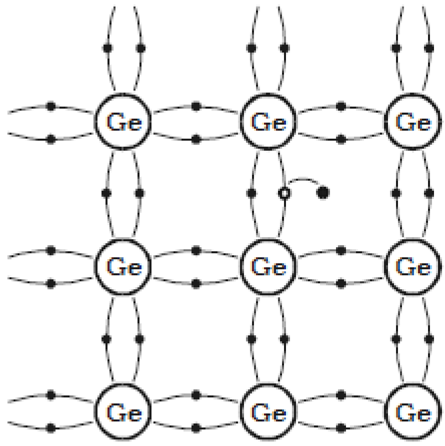
A semiconductor which is pure and contains no impurity is known as an intrinsic semiconductor. In an intrinsic semiconductor, the number of free electrons and holes are equal. Common examples of intrinsic semiconductors are pure germanium and silicon

Fig a and Fig b represent charge carriers at absolute zero temperature and at room temperature respectively.



The electrons in an intrinsic semiconductor, which move in to the conduction band at high temperatures are called as intrinsic carriers. In the valence band, a vacancy is created at the place where the electron was present, before it had moved in to the conduction band. This vacancy is called hole.

Fig c helps in understanding the creation of a hole. Consider the case of pure germanium crystal. It has four electrons in its outer or valence orbit. These electrons are known as valence electrons. When two atoms of germanium are brought close to each other, a covalent bond is formed between the atoms. If some additional energy is received, one of the electrons contributing to a covalent bond breaks and it is free to move in the crystal lattice



While coming out of the bond, a hole is said to be created at its place, which is usually represented by a open circle. The hole behaves as an *apparent free particle* with effective positive charge. An electron from the neighboring atom can break the covalent bond and can occupy this hole, creating a hole at another place. Since an electron has a unit negative charge, the hole is associated with a unit positive charge. The importance of hole is that, it may serve as a carrier of electricity in the same manner as the free electron, but in the opposite direction.

In intrinsic semiconductors, the number of free electrons, n_e is equal to the number of holes, n_h. That is $n_e = n_h = n_i$

where n_i is called intrinsic carrier concentration

Under the action of an electric field, these holes move towards negative potential giving the hole current, I_h. The total current, I is thus the sum of the electron current I_e and the hole current I_h:

$I = I_e + I_h$

It may be noted that apart from the *process of generation* of conduction electrons and holes, a simultaneous *process of recombination* occurs in which the electrons *recombine* with the holes. At equilibrium, the rate of generation is equal to the rate of recombination of charge carriers. The recombination occurs due to an electron colliding with a hole.

EXTRINSIC SEMICONDUCTOR

Electrons and holes can be generated in a semiconductor crystal with heat energy or light energy. But in these cases, the conductivity remains very low. The efficient and convenient method of generating free electrons and holes is to add very small amount of selected impurity inside the crystal. The impurity to be added is of the order of 100 ppm (parts per million). The process of addition of a very small amount of impurity into an intrinsic semiconductor is called doping.

The impurity atoms are called dopants. The semiconductor containing impurity atoms is known as impure or doped or extrinsic semiconductor.

There are three different methods of doping a semiconductor.

(i) The impurity atoms are added to the semiconductor in its molten state.

(ii) The pure semiconductor is bombarded by ions of impurity atoms.

(iii) When the semiconductor crystal containing the impurity atoms is heated, the impurity atoms diffuse into the hot crystal.

Usually, the doping material is either pentavalent atoms (bismuth, antimony, phosphorous, arsenic which have five valence electrons) or trivalent atoms (aluminium, gallium, indium, boron which have three valence electrons).

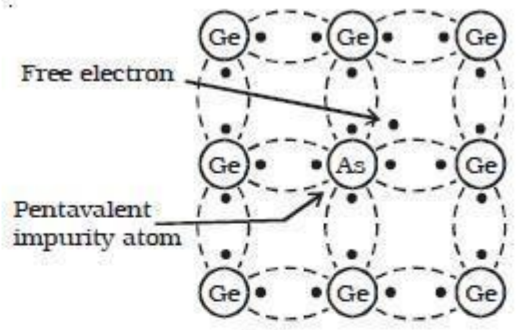
The pentavalent doping atom is known as donor atom, since it donates one electron to the conduction band of pure semiconductor.

The trivalent atom is called an acceptor atom, because it accepts one electron from the pure semiconductor atom.

Depending upon the type of impurity atoms added, an extrinsic semiconductor can be classified as N-type or P-type.

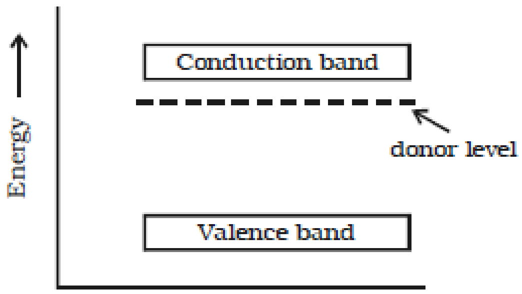
(A) N-TYPE SEMICONDUCTOR

When a small amount of pentavalent impurity such as arsenic is added to a pure germanium semiconductor crystal, the resulting crystal is called N-type semiconductor. Fig a shows the crystal structure obtained when pentavalent arsenic impurity is added with pure germanium crystal



(a)

The four valence electrons of arsenic atom form covalent bonds with electrons of neighboring four germanium atoms. The fifth electron of arsenic atom is loosely bound. This electron can move about almost as freely as an electron in a conductor and hence it will be the carrier of current. In the energy band picture, the energy state corresponding to the fifth valence electron is in the forbidden gap and lies slightly below the conduction band (Figb).



This level is known as the donor level. When the fifth valence electron is transferred to the conduction band, the arsenic atom becomes positively charged immobile ion. Each impurity atom donates one free electron to the semiconductor. These impurity atoms are called donors.

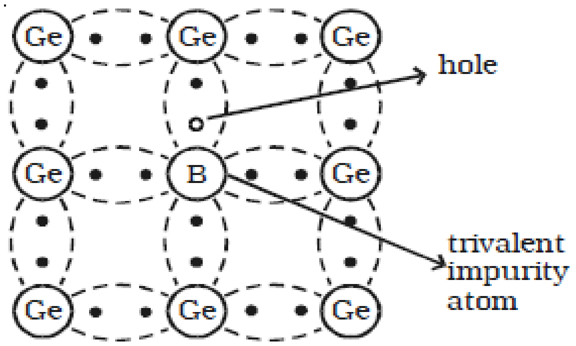
In N-type semiconductor material, the number of electrons increases, compared to the available number of charge carriers in the intrinsic semiconductor. This is because, the available larger number of electrons increases the rate of recombination of electrons with holes. Hence, in N-type semiconductor, free electrons are the majority charge carriers and holes are the minority charge carriers in an extrinsic

therefore, known as n-*type semiconductors*. For n-type semiconductors,

we have, $n_e >> n_h$

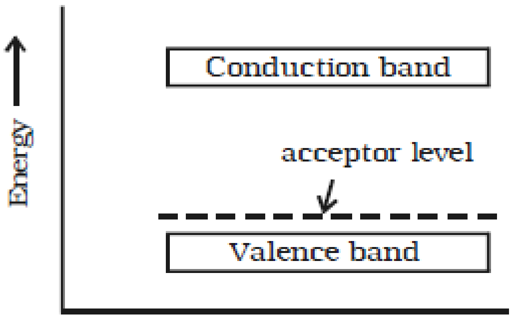
(b) P-type semiconductor

When a small amount of trivalent impurity (such as indium, boron or gallium) is added to a pure semiconductor crystal, the resulting semiconductor crystal is called P-type semiconductor. (Fig a)shows the crystal structure obtained, when trivalent boron impurity is added with pure germanium crystal.



The three valence electrons of the boron atom form covalent bonds with valence electrons of three neighborhood germanium atoms. In the fourth covalent bond, only one valence electron is available from germanium atom and there is deficiency of one electron which is called as a hole.

Hence for each boron atom added, one hole is created. Since the holes can accept electrons from neighborhood, the impurity is called acceptor. The hole, may be filled by the electron from a neighboring atom, creating a hole in that position from where the electron moves. This process continues and the hole moves about in a random manner due to thermal effects. Since the hole is associated with a positive charge moving from one position to another, this is called as P-type semiconductor. In the P-type semiconductor, the acceptor impurity produces an energy level just above the valence band. (Fig b).



Since, the energy difference between acceptor energy level and the valence band is much smaller, electrons from the valence band can easily jump into the acceptor level by thermal agitation. In P-type semiconductors, holes are the majority charge carriers and free electrons are the minority charge carriers.

Therefore, extrinsic semiconductors doped with trivalent impurity are called p-*type semiconductors*. For p-type semiconductors, the recombination process will reduce the number (*ni*)of intrinsically generated electrons to n_e. We have, for p-type semiconductors $n_h >> n_e$

Note that the crystal maintains an overall charge neutrality as the charge of additional charge carriers is just equal and opposite to that of the ionised cores in the lattice.

Conduction in p-type and n-type semiconductors

The semiconductor's energy band structure is affected by doping. In the case of extrinsic semiconductors, additional energy states due to donor impurities and acceptor impurities also exist.

In the energy band diagram of n-type Si semiconductor, the donor energy level is slightly below the bottom of the conduction band and electrons from this level move into the conduction band with very small supply of energy. At room temperature, most of the donor atoms get ionised but very few (~10–12) atoms of Si get ionised. So the conduction band will have most electrons coming from the donor impurities,

Similarly for p-type semiconductor, the acceptor energy level is slightly above the top of the valence band. With very small supply of energy an electron from the valence band can jump to the level and ionise the acceptor negatively. (Alternately, we can also say that with very small supply of energy the hole from level sinks down into the valence band. Electrons rise up and holes fall down when they gain external energy.) At room temperature, most of the acceptor atoms get ionised leaving holes in the valence band. Thus at room temperature the density of holes in the valence band is predominantly due to impurity in the extrinsic semiconductor. The electron and hole concentration in a semiconductor *in thermal equilibrium* is given by $n_e n_h = n_i^2$

Solved Numerical

Q) Suppose a pure Si crystal has 5 × 1028 atoms m^{-3}. It is doped by 1 ppm concentration of pentavalent As. Calculate the number of electrons and holes. Given that $n_i = 1.5 \times 10^{16}\ m^{-3}$.

<u>Solution:</u>

Note that thermally generated electrons ($n_i \sim 10^{16}\ m^{-3}$) are negligibly small as compared to those produced by doping.

Therefore, $n_e \approx N_D$.

Since $n_e n_h = n_i^2$

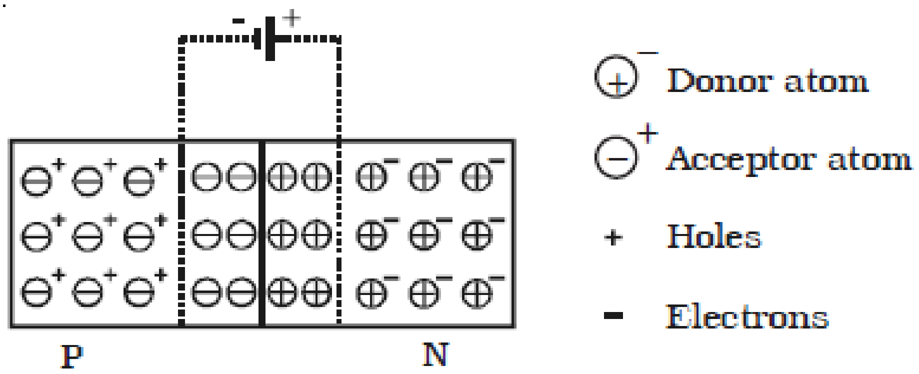
The number of holes

$n_h = (2.25 \times 10^{32})/(5 \times 10^{22})$

$n_h \sim 4.5 \times 10^9\ m^{-3}$

PN Junction diode

If one side of a single crystal of pure semiconductor (Ge or Si) is doped with acceptor impurity atoms and the other side is doped with donor impurity atoms, a PN junction is formed as shown in Fig



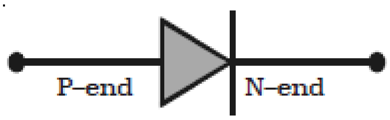
P region has a high concentration of holes and N region contains a large number of electrons. As soon as the junction is formed, free electrons and holes cross through the junction by the process of diffusion. During this process, the electrons crossing the junction from N-region into the P region, recombine with holes in the P-region very close to the junction.

Similarly holes crossing the junction from the P-region into the N-region, recombine with electrons in the N-region very close to the junction. Thus a region is formed, which does not have any mobile charges very close to the junction. This region is called depletion region. In this region, on the left side of the junction, the acceptor atoms become negative ions and on the right side of the junction, the donor atoms become positive ions

An electric field is set up, between the donor and acceptor ions in the depletion region. The potential at the N-side is higher than the potential at P-side. Therefore electrons in the N-side are prevented to go to the lower potential of P-side. Similarly, holes in the P-side find themselves at a lower potential and are prevented to cross to the N-side. Thus, there is a barrier at the junction which opposes the movement of the majority charge carriers. The difference of potential from one side of the barrier to the other side is called potential barrier. The potential barrier is approximately 0.7V for a silicon PN junction and 0.3V for a germanium PN junction. The distance from one side of the barrier to the other side is called the width of the barrier, which depends upon the nature of the material.

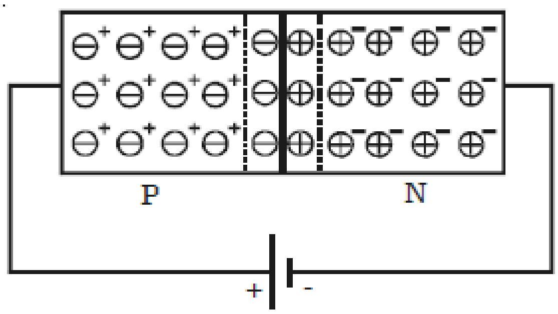
Symbol for a semiconductor diode

The diode symbol is shown in Fig The P-type and N-type regions are referred to as P–end and N–end respectively. The arrow on the diode points the direction of



Forward biased PN junction diode

When the positive terminal of the battery is connected to P-side and negative terminal to the N-side, so that the electric field across diode due to battery is in opposite direction to the electric field of barrier, then the PN junction diode is said to be forward biased.

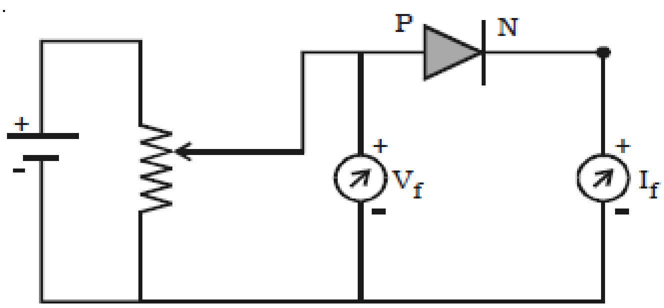


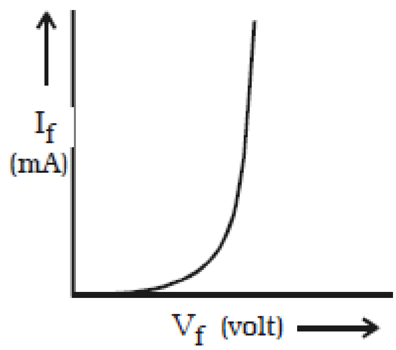
When the PN junction is forward biased (Fig), the applied positive potential repels the holes in the P-region, and the applied negative potential repels the electrons in the N-region, so the charges move towards the junction.

If the applied potential difference is more than the potential barrier, some holes and free electrons enter the depletion region.

Hence, the potential barrier as well as the width of the depletion region are reduced. The positive donor ions and negative acceptor ions within the depletion region regain electrons and holes respectively. As a result of this, the depletion region disappears and the potential barrier also disappears. Hence, under the action of the forward potential difference, the majority charge carriers flow across the junction in opposite direction and constitute current flow in the forward direction.

Forward bias characteristics





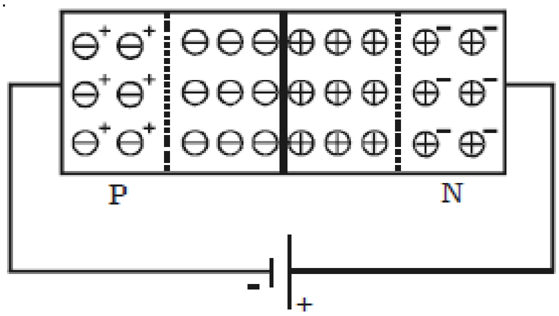
The circuit for the study of forward bias characteristics of PN junction diode is shown in Fig The voltage between P–end and N–end is increased from zero in suitable equal steps and the corresponding currents are noted down. (Fig b) shows the forward bias characteristic curve of the diode. Voltage is the independent variable. Therefore, it is plotted along X–axis. Since, current is the dependent variable, it is plotted against Y–axis. From the characteristic curve, the following conclusions can be made.

(i) The forward characteristic is not a straight line. Hence the ratio V/I is not a constant (i.e) the diode does not obey Ohm’s law. This implies that the semiconductor diode is a non-linear conductor of electricity.

(ii) It can be seen from the characteristic curve that initially, the current is very small. This is because, the diode will start conducting, only when the external voltage overcomes the barrier potential (0.7V for silicon diode). As the voltage is increased to 0.7 V, large number of free electrons and holes start crossing the junction. Above 0.7V, the current increases rapidly. The voltage at which the current starts to increase rapidly is known as cut-in voltage or knee voltage of the diode

Reverse biased PN junction diode

When the positive terminal of the battery is connected to the difference is in the same direction as that of barrier potential, the junction is said to be reverse biased.



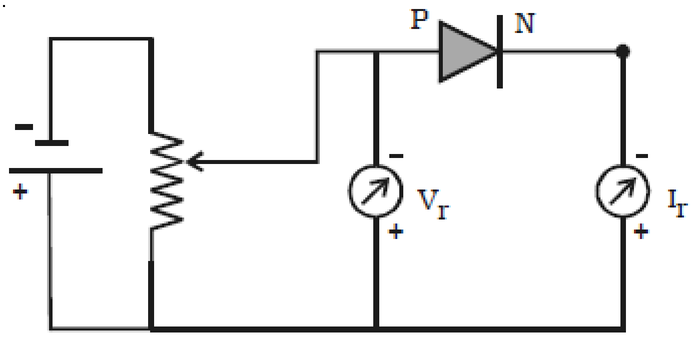
When the PN junction is reverse biased (Fig), electrons in the N region and holes in the P-region are attracted away from the junction N-side and negative terminal to the P-side, so that the applied potential Because of this, the number of negative ions in the P-region and positive ions in the N-region increases. Hence the depletion region becomes wider and the potential barrier is increased.

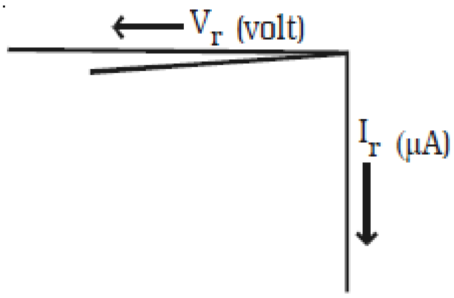
Since the depletion region does not contain majority charge carriers, it acts like an insulator. Therefore, no current should flow in the external circuit. But, in practice, a very small current of the order of few microamperes flows in the reverse direction. This is due to the minority carriers flowing in the opposite direction. This reverse current is small, because the number of minority carriers in both regions is very small. Since the major source of minority carriers is, thermally broken covalent bonds, the reverse current mainly depends on the junction temperature.

Reverse bias characteristics

The circuit for reverse bias characteristics of PN junction diode is shown(Fig.)

The voltage is increased from zero in suitable steps. For each voltage, the corresponding current readings are noted down. Fig b shows the reverse bias characteristic curve of the diode. From the characteristic curve, it can be concluded that, as voltage is increased from zero, reverse current (in the order of microamperes) increases and reaches the maximum value at a small value of the reverse voltage. When the voltage is further increased, the current is almost independent of the reverse voltage upto a certain critical value.
This reverse current is known as the reverse saturation current or leakage current. This current is due to the minority charge carriers, which depends on junction temperature.



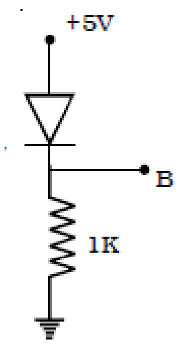


Avalanche breakdown :

When both sides of the PN junction are lightly doped and the depletion layer becomes large, avalanche breakdown takes place. In this case, the electric field across the depletion layer is not so strong. The minority carriers accelerated by the field, collide with the semiconductor atoms in the crystal.
Because of this collision with valence electrons, covalent bonds are broken and electron hole pairs are generated. These charge carriers, so produced acquire energy from the applied potential and in turn produce more and more carriers. This cumulative process is called avalanche multiplication and the breakdown is called avalanche breakdown.

Solved Numerical

Q) Find the voltage at the point B in the figure (Silicon diode is used).

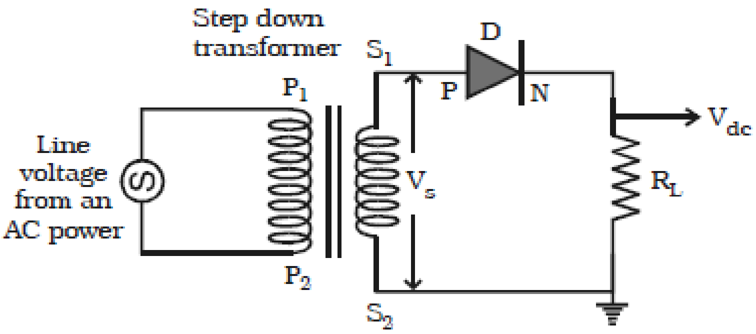


Solution:
The potential drop across the diode is equal to the knee voltage when diode is in forward biases. This voltage for Si diode is 0.7V

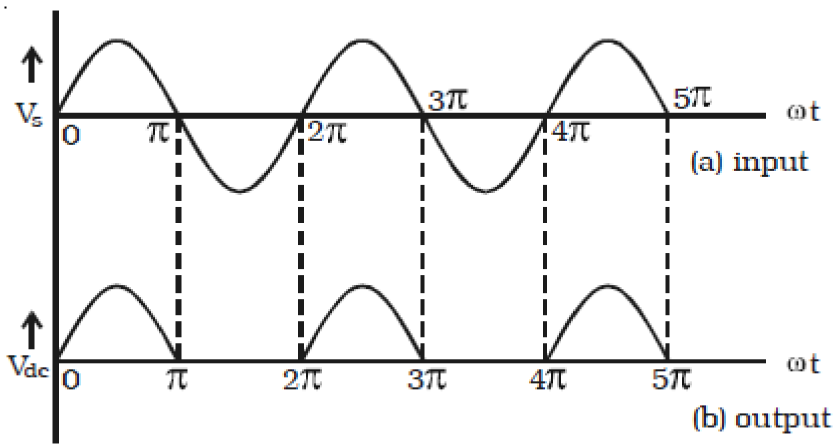
Now by applying Kirchhoff law
$5 = 0.7 + V_R$
$V_R = 4.3$ V
Now V_R = potential at B
Thus potential at B is 4.3V

Half wave rectifier

A circuit which rectifies half of the a.c wave is called half wave rectifier. Fig shows the circuit for half wave rectification. The a.c. voltage (Vs) to be rectified is obtained across the secondary ends S_1 S_2 of the transformer.

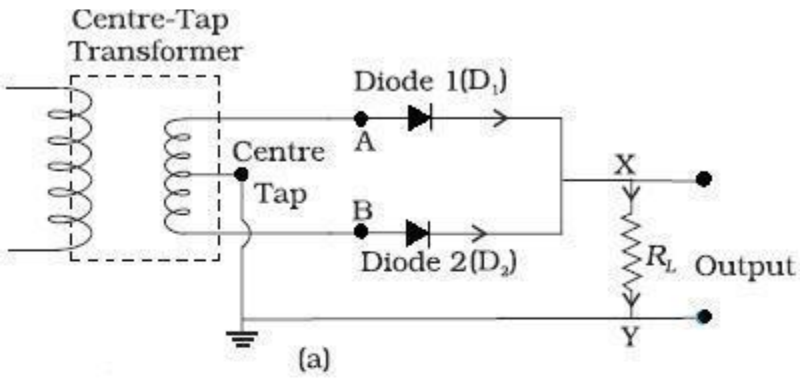


The P-end of the diode D is connected to S_1 of the secondary coil of the transformer. The N-end of the diode is connected to the other end S_2 of the secondary coil of the transformer, through a load resistance R_L.
The rectified output voltage V_{dc} appears across the load resistance R_L. During the positive half cycle of the input a.c. voltage Vs, S1 will be positive and the diode is forward biased and hence it conducts.
Therefore, current flows through the circuit and there is a voltage drop across RL. This gives the output voltage as shown in Fig. During the negative half cycle of the input a.c. voltage (Vs), S_1 will be negative and the diode D is reverse biased. Hence the diode does not conduct. No current flows through the circuit and the voltage drop across R_L will be zero. Hence no output voltage is obtained.
Thus corresponding to an alternating input signal, unidirectional pulsating output is obtained. The ratio of d.c. power output to the a.c. power input is known as rectifier efficiency. The efficiency of half wave rectifier is approximately 40.6%.

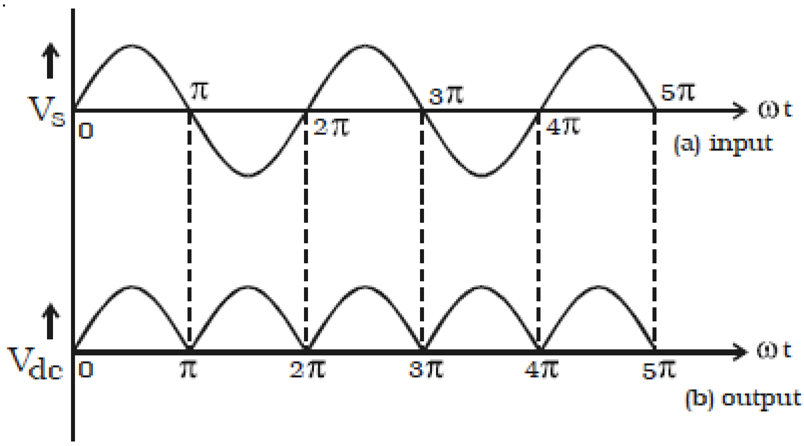


Full-wave rectifier:

The circuit using two diodes, shown in Fig. (a), gives output rectified voltage corresponding to both the positive as well as negative half of the ac cycle. Hence, it is known as *full-wave rectifier*.
Here the p-side of the two diodes are connected to the ends of the secondary of the transformer. The n-side of the diodes are connected together and the output is taken between this common point of diodes and the midpoint of the secondary of the transformer. So for a full-wave rectifier the secondary of the transformer is provided with a centre tapping and so it is called *centre-tap transformer*.
Suppose the input voltage to A with respect to the centre tap at any instant is positive. It is clear that, at that instant, voltage at B being out of phase will be negative as shown in So, diode D_1 gets forward biased and conducts (while D2 being reverse biased is not conducting). Current flows through path AD1XY to central tapping.
Hence, during this positive half cycle we get an output current (and a output voltage across the load resistor R_L) as shown in (Fig.b).
In the course of the ac cycle , the voltage at B would be positive. In this part of the cycle diode D_1 would not conduct but diode D_2 would, giving an output current path of current will be BD_2 XY to central tapping and output voltage (across R_L) during the negative half cycle of the input ac.
Thus, we get output voltage during both the positive as well as the negative half of the cycle. This is a more efficient circuit for getting rectified voltage or current than the halfwave rectifier
The rectified voltage is in the form of pulses of the shape of half sinusoids. Though it is unidirectional it does not have a steady value.



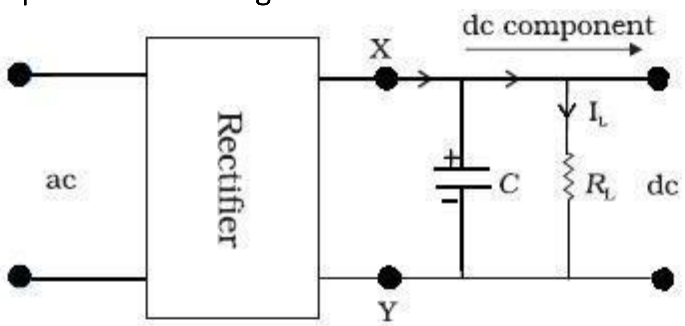
To get steady dc output from the pulsating voltage normally a capacitor is connected across the output terminals (parallel to the load R_L). One can also use an inductor in series with *RL* for the same purpose. Since these additional circuits appear to *filter* out the *ac ripple* and give a *pure dc* voltage, so they are called filters.



Filter circuits and regulation property of the power supply

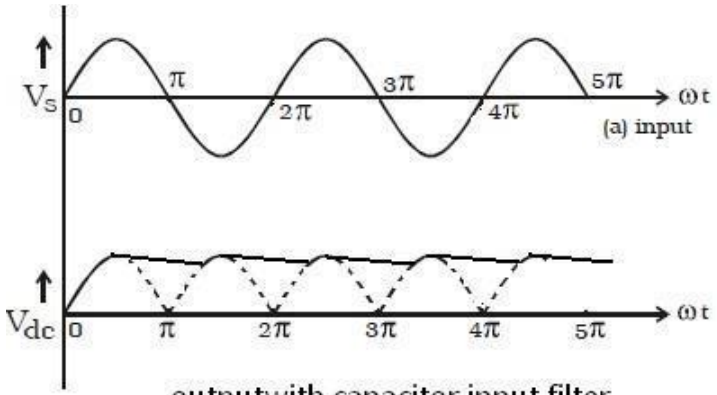
The rectified voltage is in the form of pulses of the shape of half sinusoids. Though it is unidirectional it does not have a steady value.

To get steady dc output from the pulsating voltage normally a capacitor is connected across the output terminals (parallel to the load R_L). One can also use an inductor in series with R_L for the same purpose. Since these additional circuits appear to *filter* out the *ac ripple* and give a *pure dc* voltage, so they are called filters. Now we shall discuss the role of capacitor in filtering.



When the voltage across the capacitor is rising, it gets charged. If there is no external load, it remains charged to the peak voltage of the rectified output.

When there is a load, it gets discharged through the load and the voltage across it begins to fall. In the next half-cycle of rectified output it again gets charged to the peak value (Fig.).



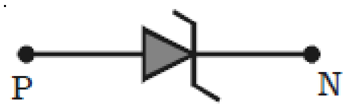
outputwith capacitor input filter

The rate of fall of the voltage across the capacitor depends upon the inverse product of capacitor *C* and the effective resistance R_L used in the circuit and is called the *time constant*.

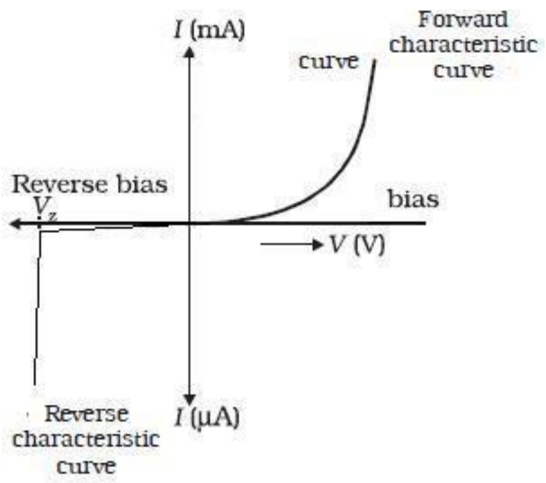
To make the time constant large value of *C* should be large. So capacitor input filters use large capacitors. The *output voltage* obtained by using capacitor input filter is nearer to the *peak voltage* of the rectified voltage. This type of filter is most widely used in power supplies.

Zener diode

It is a special purpose semiconductor diode, named after its inventor C. Zener.
It is designed to operate under reverse bias in the breakdown region and used as a voltage regulator. The symbol for Zener diode is shown in (a).



Zener diode is fabricated by heavily doping both p-, and n- sides of the junction.
Due to this, depletion region formed is very thin ($<10^{-6}$ m) and the electric field of the junction is extremely high (~5×106 V/m) even for a small reverse bias voltage of about 5V. The I-V characteristics of a Zener diode is shown in Fig.



It is seen that when the applied reverse bias voltage(*V*) reaches the breakdown voltage (V_z) of the Zener diode, there is a large change in the current. Note that after the breakdown voltage *Vz*, a large change in the current can be produced by almost insignificant change in the reverse bias voltage. In other words, Zener voltage remains constant, even though current through the Zener diode varies over a wide range. This property of the Zener diode is used for regulating supply voltages so that they are constant.

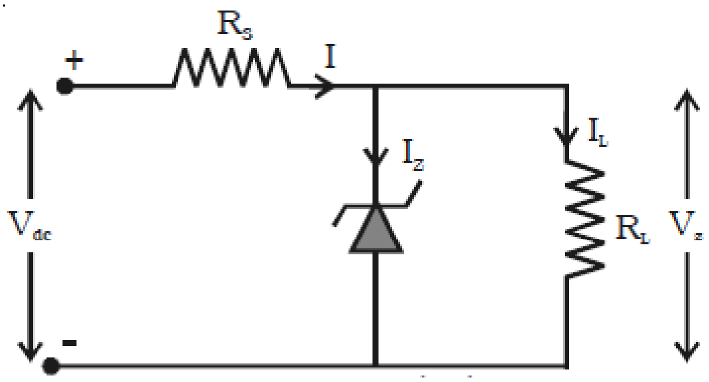
Zener breakdown :

We know that reverse current is due to the flow of electrons (minority carriers) from p → n and holes from n → p.
When both sides of the PN junction are heavily doped, consequently the depletion layer is narrow. Zener breakdown takes place in such a thin narrow junction
As the reverse bias voltage is increased, the electric field at the junction becomes significant. When the reverse bias voltage V = V_z, then the electric field strength is high enough to pull valence electrons from the host atoms on the p-side which are accelerated These electrons account for high current observed at the breakdown. The emission of electrons from the host atoms due to the high electric field is known as internal field emission or field ionisation. The electric field required for field ionisation is of the order of 10^6 V/m.

Zener diode as voltage regulator:

To maintain a constant voltage across the load, even if the input voltage or load current varies, voltage regulation is to be made.

A Zener diode working in the break down region can act as voltage regulator. The circuit in which a Zener diode is used for maintaining a constant voltage across the load R_L is shown in Fig

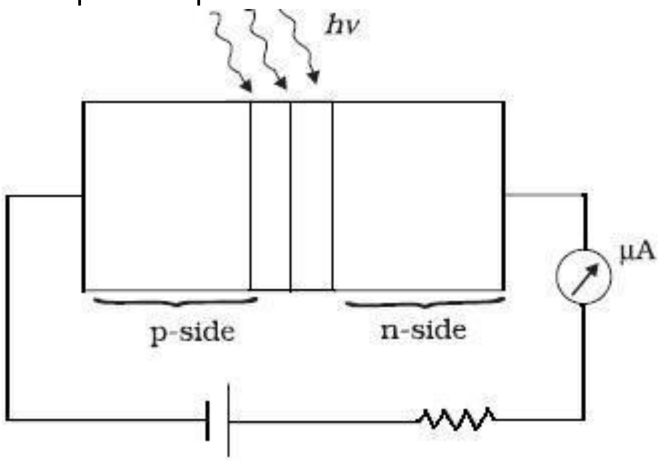


The Zener diode in reverse biased condition is connected in parallel with the load R_L. Let V_{dc} be the unregulated dc voltage and V_z be Zener voltage (regulated output voltage).
R_s is the current limiting resistor. It is chosen in such a way that the diode operates in the breakdown region. Inspite of changes in the load current or in the input voltage, the Zener diode maintains a constant voltage across the load. The action of the circuit can be explained as given below.
(i) load current varies, input voltage is constant : Let us consider that the load current increases. Zener current hence decreases, and the current through the resistance Rs is a constant.
The output voltage is $V_z = V_{dc} – IR_s$, since the total current I remains constant, output voltage remains constant.
(ii) input voltage varies :
If input voltage increases, In the breakdown region, Zener voltage remains constant even though the current through the Zener diode changes.
Similarly, if the input voltage decreases, the current through *Rs* and Zener diode also decreases. The voltage drop across *Rs* decreases without any change in the voltage across the Zener diode.
Thus any increase/decrease in the input voltage results in, increase/decrease of the voltage drop across *Rs* without any change in voltage across the Zener diode. Thus the Zener diode acts as a voltage regulator.
We have to select the Zener diode according to the required output voltage and accordingly the series resistance *Rs*.

(i) *Photodiode*

A Photodiode is again a special purpose p-n junction diode fabricated with a transparent window to allow light to fall on the diode.
It is operated under reverse bias.
Reverse saturation current flows through the PN junction diode on connecting it in a reverse bias mode.
The reverse saturation current can be increased by making more light incident on it

When the photodiode is illuminated with light (photons) with energy ($h\nu$) greater than the energy gap (E_g) of the semiconductor, then electron-hole pairs are generated due to the absorption of photons.



(a)

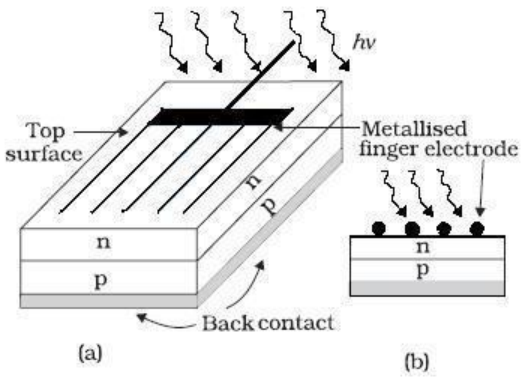
The diode is fabricated such that the generation of *e-h* pairs takes place in or near the depletion region of the diode.
Due to electric field of the junction, electrons and holes are separated before they recombine. The direction of the electric field is such that electrons reach n-side and holes reach p-side.
When an external load is connected ,electrons are collected on n-side and holes are collected on p-side giving rise to reverse saturation current.
The magnitude of the photocurrent depends on the intensity of incident light photocurrent is proportional to incident light intensity).
Thus photodiode can be used as a photo detector to detect optical signals.

Light emitting diode

It is a heavily doped p-n junction which under forward bias emits spontaneous radiation.
The diode is encapsulated with a transparent cover so that emitted light can come out.
When the diode is forward biased, electrons are sent from n → p and holes are sent from p → n.
At the junction boundary the concentration of minority carriers increases compared to the equilibrium concentration (i.e., when there is no bias).
Thus at the junction boundary on either side of the junction, excess minority carriers are there which recombine with majority carriers near the junction.
On recombination, the energy is released in the form of photons. Photons with energy equal to or slightly less than the band gap are emitted.
When the forward current of the diode is small, the intensity of light emitted is small. As the forward current increases, intensity of light increases and reaches a maximum. Further increase in the forward current results in decrease of light intensity.
LEDs are biased such that the light emitting efficiency is maximum.

The *V-I* characteristics of a LED is similar to that of a Si junction diode. But the threshold voltages are much higher and slightly different for each colour.
The reverse breakdown voltages of LEDs are very low, typically around 5V. So care should be taken that high reverse voltages do not appear across them.
LEDs that can emit red, yellow, orange, green and blue light are commercially available.
The semiconductor used for fabrication of visible LEDs must at least have a band gap of 1.8 eV (spectral range of visible light is from about 0.4 µm to 0.7 µm, i.e., from about 3 eV to 1.8 eV).
The compound semiconductor Gallium Arsenide – Phosphide ($GaAs_{1-x}P_x$) is used for making LEDs of different colours.
$GaAs_{0.6}P_{0.4}$ (E_g ~ 1.9 eV) is used for red LED.
GaAs (E_g ~ 1.4 eV) is used for making infrared LED.
These LEDs find extensive use in remote controls, burglar alarm systems, optical communication, etc. Extensive research is being done for developing white LEDs which can replace incandescent lamps.
LEDs have the following advantages over conventional incandescent low power lamps:
(i) Low operational voltage and less power.
(ii) Fast action and no warm-up time required.
(iii) The bandwidth of emitted light is 100 Å to 500 Å or in other words it is nearly (but not exactly) monochromatic.
(iv) Long life and ruggedness.
(v) Fast on-off switching capability.

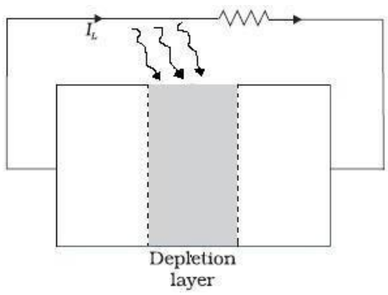
Solar cell



A solar cell is basically a p-n junction which generates emf when solar radiation falls on the p-n junction. It works on the same principle (photovoltaic effect) as the photodiode, except that no external bias is applied and the junction area is kept much larger for solar radiation to be incident because we are interested in more power. A simple p-n junction solar cell is shown in figure
A p-Si wafer of about 300 µm is taken over which a thin layer (~0.3 µm) of n-Si is grown on

one-side by diffusion process. The other side of p-Si is coated with a metal (back contact). On the top of n-Si layer, metal finger electrode (or metallic grid) is deposited. This acts as a front contact. The metallic grid occupies only a very small fraction of the cell area (<15%) so that light can be incident on the cell from the top.

The generation of emf by a solar cell, when light falls on, it is due to the following three basic processes: generation, separation and collection—

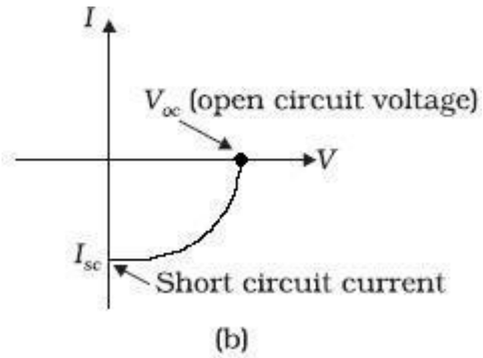


(i) generation of e-h pairs due to light (with $h\nu > Eg$) close to the junction;

(ii) separation of electrons and holes due to electric field of the depletion region. Electrons are swept to n-side and holes to p-side;

(iii) the electrons reaching the n-side are collected by the front contact and holes reaching p-side are collected by the back contact. Thus p-side becomes positive and n-side becomes negative giving rise to *photovoltage*.

When an external load is connected as shown in the Fig. a photocurrent I_L flows through the load. A typical *I-V* characteristics of a solar cell is shown in the Fig.b



(b)

Note that the *I* – *V* characteristics of solar cell is drawn in the fourth quadrant of the coordinate axes. This is because a solar cell does not draw current but supplies the same to the load.

Semiconductors with band gap close to 1.5 eV are ideal materials for solar cell fabrication. Solar cells are made with semiconductors like

Si (Eg = 1.1 eV),

GaAs (Eg = 1.43 eV),

CdTe (Eg = 1.45 eV),

$CuInSe_2$ (Eg = 1.04eV), etc.

The important criteria for the selection of a material for solar cell fabrication are

(i) Band gap (~1.0 to 1.8 eV),

(ii) High optical absorption (~104 cm–1), electrical conductivity,

(iv) Availability of the raw material, and

(v) Cost. Note that sunlight is not always required for a solar cell.

Any light with photon energies greater than the band gap will do. Solar cells are used to power electronic devices in satellites and space vehicles and also as power supply to some calculators.

Junction transistor

A junction transistor is a solid state device. It consists of silicon or germanium crystal containing two PN junctions.

The two PN junctions are formed between the three layers. These are called base, emitter and collector.

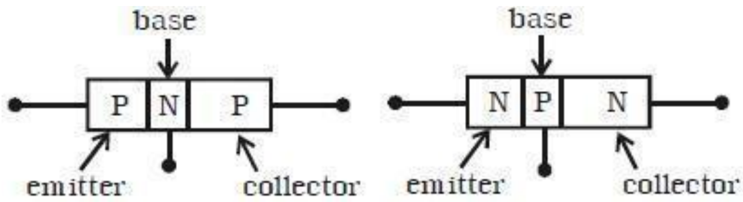
(i) Base (B) layer : It is a very thin layer, the thickness is about 25 microns. It is the central region of the transistor.

(ii) Emitter (E) and Collector (C) layers : The two layers on the opposite sides of B layer are emitter and collector layers. They are of the same type of the semiconductor.

An ohmic contact is made to each of these layers. The junction between emitter and base is called emitter junction. The junction between collector and base is called collector junction.

In a transistor, the emitter region is heavily doped, since emitter has to supply majority carriers. The base is lightly doped. The collector region is lightly doped. Since it has to accept majority charge carriers, it is physically larger in size. Hence, emitter and collector cannot be interchanged.

The construction of PNP and NPN transistors are shown in Fig a and Fig b respectively.



(a) *Construction of PNP transistor* (b) *Construction of NPN transistor*

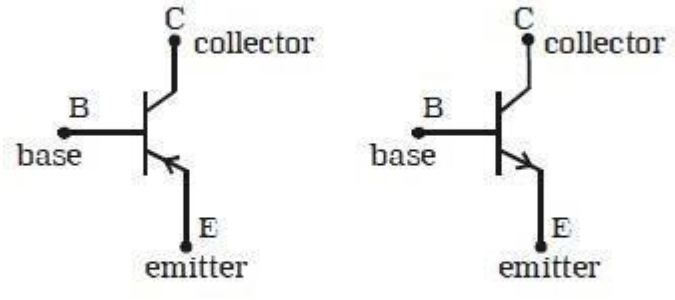
For a transistor to work, the biasing to be given are as follows :

(i) The emitter-base junction is forward biased, so that majority charge carriers are repelled from the emitter and the junction offers very low resistance to the current.

(ii) The collector-base junction is reverse biased, so that it attracts majority charge carriers and this junction offers a high resistance to the current.

Transistor circuit symbols

The circuit symbols for a PNP and NPN transistors are shown in Fig

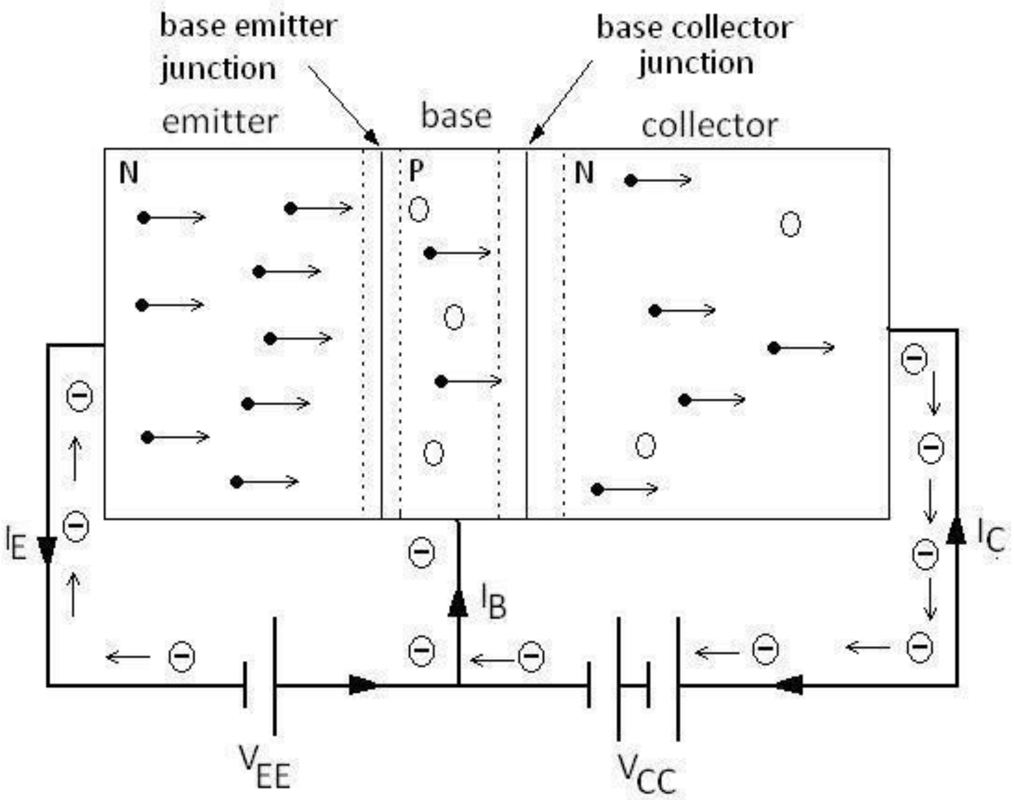


(a) PNP (b) NPN

The arrow on the emitter lead pointing towards the base represents a PNP transistor
When the emitter-base junction of a PNP transistor is forward biased, the direction of the conventional current flow is from emitter to base.
NPN transistor is represented by arrow on the emitter lead pointing away from the base.
When the emitter base junction of a NPN transistor is forward biased, the direction of the conventional current is from base to emitter.

Working of a NPN transistor

A NPN transistor is like two PN junction diodes, which are placed back-to-back. At each junction, there is a depletion region which gives rise to a potential barrier. The external biasing of the junction is provided by the batteries V_{EE} and V_{CC} as shown in Fig.
The emitter base junction is forward biased and the collector base junction is reverse biased.
Since the emitter-base junction is forward biased, a large number of electrons cross the junction and enters the base constitutes current I_E.
The base width is small and has fewer concentration of impurity as a result only 5% of electrons entering base recombine with holes.
The rest of the electrons enters collector region due to influence of the battery V_{CC}.
For every electron entering the collector one electron flows in the external circuit and constitutes the collector current I_C.
Similarly for every electron combing with the hole in the base section, there is one electron which gets attracted by V_{EE} and flows as base current I_B in external circuit
Applying Kirchoff's current law to the circuit, the emitter current is the sum of collector current and base current. $I_E = I_C + I_B$



This equation is the fundamental relation between the currents in a transistor circuit. This equation is true regardless of transistor type or transistor configuration. The action of PNP transistor is similar to that of NPN transistor.

Solved Numerical

Q) In NPN transistor about 10^{10} electrons enter the emitter in 1μs when it is connected to a battery. About 2% electrons recombine with the holes in the base. Calculate the values of I_E , I_B, I_C, α_{dc}, and β_{dc} (e = 1.6 ×10^{-19} C)

Solution:

As per the definition of current

$$I_E = \frac{Q}{t} = \frac{ne}{t}$$

$$I_E = \frac{10^{10} \times 1.6 \times 10^{-19}}{10^{-6}} = 1600\mu A$$

2% of the total current entering the base from the emitter recombine with the holes which constitutes the base current I_B. The rest of the 98% electrons reaches the collector and constitutes the collector current

I_B = 0.021, I_E = 0.02 × 1600 = 32 μA

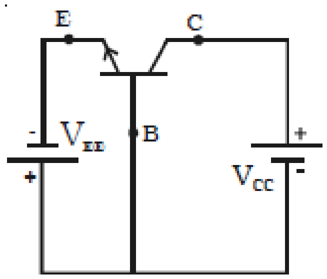
I_C = 0.98, I_E = 0.98 × 1600 =1568 μA

$$\alpha_{dc} = \frac{I_C}{I_E} = \frac{1568 \times 10^{-6}}{1600 \times 10^{-6}} = 0.98$$

$$\beta_{dc} = \frac{I_C}{I_B} = \frac{1568 \times 10^{-6}}{32 \times 10^{-6}} = 49$$

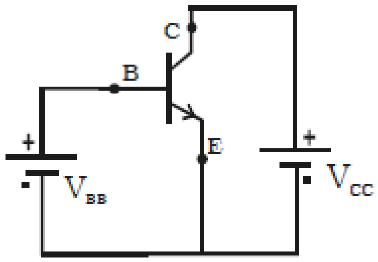
Transistor circuit configurations

There are three types of circuit connections (called configurations or modes) for operating a transistor. They are (i) common base (CB) mode



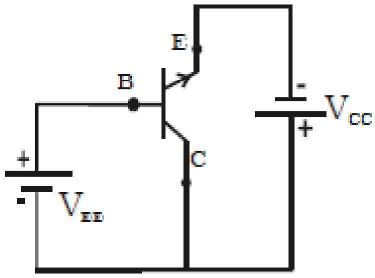
(a) CB mode

(ii) common emitter (CE) mode



(b) CE mode

(iii) common collector (CC) mode.



(c) CC mode

In a similar way, three configurations can be drawn for PNP transistor.

Current amplification factors α and β and the relation between them

The current amplification factor or current gain of a transistor is the ratio of output current to the input current.

If the transistor is connected in common base mode, the current gain α

$$\alpha = \frac{I_C}{I_E}$$

and if the transistor is connected in common emitter mode, the current gain β

$$\beta = \frac{I_C}{I_B}$$

Since 95% of the injected electrons reach the collector, the collector current is almost equal to the emitter current. Almost all transistors have α , in the range 0.95 to 0.99
We know that

$$\alpha = \frac{I_C}{I_E}$$

And $I_E = I_C + I_B$ Thus

$$\alpha = \frac{I_C}{I_C + I_B}$$

$$\frac{1}{\alpha} = \frac{I_C + I_B}{I_C} = 1 + \frac{I_B}{I_C}$$

$$\frac{1}{\alpha} - 1 = \frac{I_B}{I_C}$$

$$\frac{1}{\alpha} - 1 = \frac{1}{\beta}$$

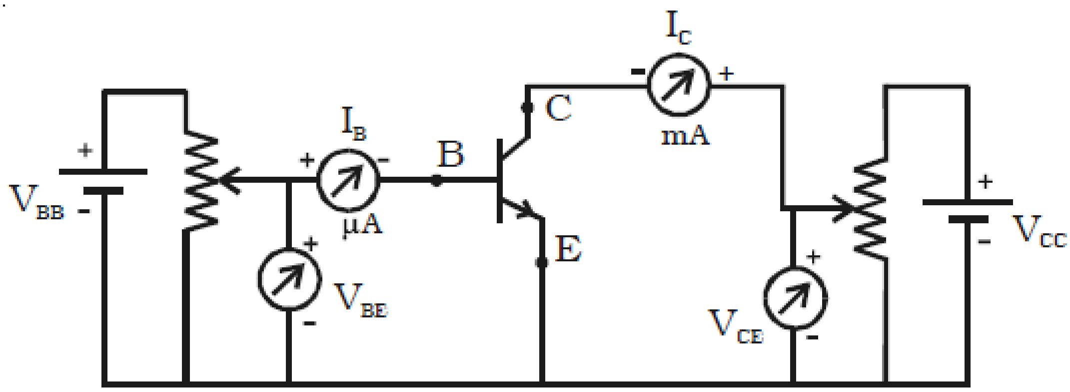
$$\beta = \frac{\alpha}{1 - \alpha}$$

Usually β lies between 50 and 300. Some transistors have β as high as 1000.
Similarly we can prove

$$\alpha = \frac{\beta}{1 + \beta}$$

Characteristics of an NPN transistor in common emitter configuration

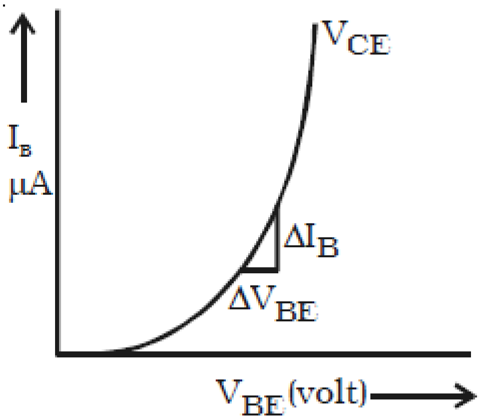
The three important characteristics of a transistor in any mode
are (i) input characteristics (ii) output characteristics and (iii) transfer characteristics.
The circuit to study the characteristic curves of NPN transistor in common emitter mode is as shown in Fig



Input characteristics

Input characteristic curve is drawn between the base current (I_B) and voltage between base and emitter (V_{BE}), when the voltage between collector and emitter (V_{CE}) is kept constant at a particular value. V_{BE} is increased in suitable equal steps and corresponding base current is noted. The procedure is repeated for different values of V_{CE}.

I_B values are plotted against V_{BE} for constant V_{CE}. The input characteristic thus obtained is shown in Fig



The input impedance of the transistor is defined as the ratio of small change in base – emitter voltage to the corresponding change in base current at a given V_{CE}.

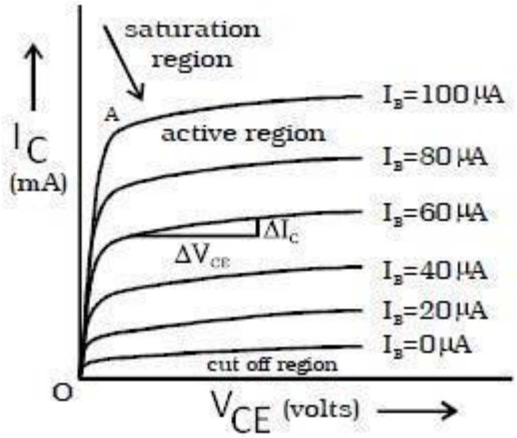
Input impedance, r_i

$$r_i = \left(\frac{\Delta V_{BE}}{\Delta I_B}\right)_{V_{CE}}$$

The input impedance of the transistor in CE mode is very high.

Output characteristics

Output characteristic curves are drawn between I_C and V_{CE}, when I_B is kept constant at a particular value. The base current I_B is kept at a constant value, by adjusting the base emitter voltage V_{BE}. V_{CE} is increased in suitable equal steps and the corresponding collector current is noted. The procedure is repeated for different values of I_B. Now, Ic versus V_{CE} curves are drawn for different values of I_B. The output characteristics thus obtained are represented in Fig.



The three regions of the characteristics can be discussed as follows :
Saturation region *:*
The initial part of the curve (ohmic region, OA) is called saturation region. (i.e) The region in between the origin and knee point. (Knee point is the point, where I_c is about to become a constant). In this region both base-emitter region and base-collector region are forward bias.

Cut off region :

There is very small collector current in the transistor, even when the base current is zero ($I_B = 0$). In the output characteristics, the region below the curve for $I_B = 0$ is called cut off region. Below the cut off region, the transistor does not function. In this region both base-emitter region and base-collector region are reverse biased.

Active region :

The central region of the curves is called active region. In the active region, the curves are uniform. In this region, E-B junction is forward biased and C-B junction is reverse biased. The output impedance r_o is defined as the ratio of variation in the collector emitter voltage to the corresponding variation in the collector current at a constant base current in the active region of the transistor characteristic curves.

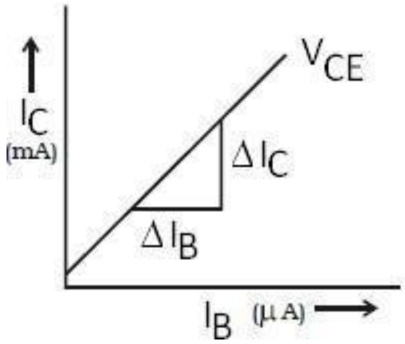
output impedance, r_o

$$r_o = \left(\frac{\Delta V_{CE}}{\Delta I_C}\right)_{I_B}$$

The output impedance of a transistor in CE mode is low.
Its value can be found out from the input characteristic curve. Normally its value is found between 50kΩ to 100kΩ

Transfer characteristics

The transfer characteristic curve is drawn between I_C and I_B, when V_{CE} is kept constant at a particular value. The base current I_B is increased in suitable steps and the collector current I_C is noted down for each value of I_B. The transfer characteristic curve is shown in Fig.



The current gain is defined as the ratio of a small change in the collector current to the corresponding change in the base current at a constant V_{CE}.
current gain, β

$$\beta = \left(\frac{\Delta I_C}{\Delta I_B}\right)_{V_{CE}}$$

The common emitter configuration has high input impedance, low output impedance and higher current gain when compared with common base configuration.
Taking the ratio of β and r_i for ac circuit

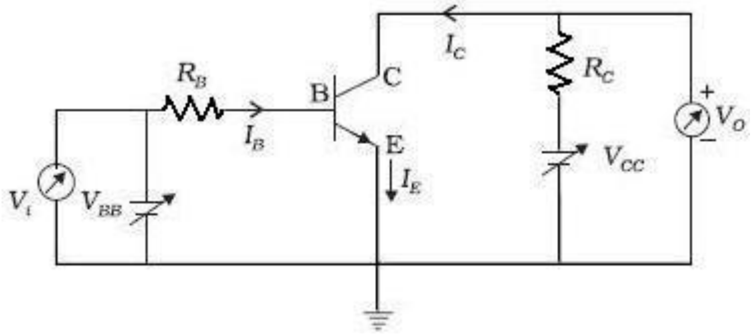
$$\frac{\beta}{r_i} = \frac{\Delta I_C / \Delta I_B}{\Delta V_{BE} / \Delta I_B} = \frac{\Delta I_C}{\Delta V_{BE}}$$

Ratio of the change in the current in the output circuit (ΔI_C) to the change in the input voltage (ΔV_{BE}) is known as the trans-conductance g_m its unit is *mho*

$$g_m = \frac{\Delta I_C}{\Delta V_{BE}} = \frac{\beta}{r_i}$$

Transistor as switch

In an ideal ON/OFF switch, when it is OFF the current is not flowing in the circuit because switch offers infinite resistance.
When switch is in On condition, maximum current flows because its resistance is zero.
We can prepare such an electronic switch by using the resistor.
The operating point switch from cutoff to saturation along the load line.



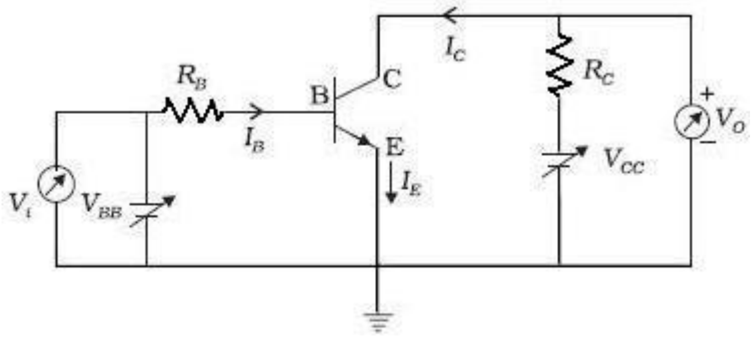
We shall try to understand the operation of the transistor as a switch by analyzing the behavior of the base-biased transistor in CE configuration as shown in Fig.
Applying Kirchhoff's voltage rule to the input and output sides of this circuit, we get
$V_{BB} = I_B R_B + V_{BE}$ and $V_{CE} = V_{CC} - I_C R_C$.
We shall treat V_{BB} as the dc input voltage V_i and V_{CE} as the dc output voltage V_O.
So, we have
$V_i = I_B R_B + V_{BE}$ ------- eq(1)and
$V_o = V_{CC} - I_C R_C$ ------- eq(2)
Let us see how V_o changes as V_i increases from zero onwards.

(i) When input voltage V_i is zero or less than 0.6V for Si transistor (transistor cut in voltage), the base current I_B will be zero. Hence the collector current will also zero $I_C = 0$
From eq(2) $V_O = V_{CC}$
In this situation resistance of output circuit is very large. Hence the current is not flowing through it. This is the 'OFF' or "cut off" condition of the transistor.

(ii) When the input voltage will be $V_i = V_{CC}$, the base current is maximum, hence the collector current is maximum. The voltage drop ($I_C R_L$) across the load resistance R_L will be approximately V_{CC} . According to eq(2)
$V_O = 0$
In this condition resistance of the output circuit of the transistor is very small to the effect that maximum current is flowing through it. This is called the "ON" condition or saturation condition of the transistor.

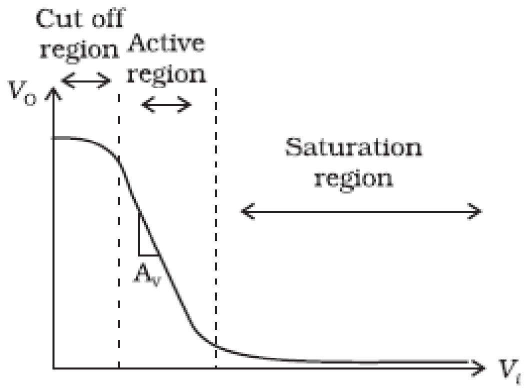
Alternatively, we can say that a *low* input to the transistor gives a *high* output and a *high* input gives a *low* output. The switching circuits are designed in such a way that the transistor does not remain in active state. This circuit is used to make 'NOT' gate in the digital electronics

Transistor as an Amplifier
The important function of a transistor is the amplification. An amplifier is a circuit capable of magnifying the amplitude of weak signals. The important parameters of an amplifier are input impedance, output impedance, current gain and voltage gain. A good design of an amplifier circuit must possess high input impedance, low output impedance and high current gain.
Transistor act as an amplifier when operated in active region.
In this region base-emitter region is forward biased and base-collector region is reverse biased.



The A.C. signal (V_i) causes the change in the base emitter voltage V_{BE}. This result in the change in the base current I_B. the change in the base current is of the order of microampere. This results in the change in the collector current equal to βI_B, which is of the order of milliampere.

For using the transistor as an amplifier we will use the active region of the V_o versus V_i curve.



The slope of the linear part of the curve represents the rate of change of the output with the input. It is negative because the output is $V_O = V_{CC} - I_C R_C$ and not $I_C R_C$. That is why as input voltage of the C_E amplifier increases its output voltage decreases and the output is said to be out of phase with the input.

If we consider ΔV_o and ΔV_i as small changes in the output and input voltages then $\Delta V_o/\Delta V_i$ is called the small signal voltage gain A_V of the amplifier.

If the V_{BB} voltage has a fixed value corresponding to the midpoint of the active region, the circuit will behave as a CE amplifier with voltage gain $\Delta V_o/ \Delta V_i$. We can express the voltage gain A_V in terms of the resistors in the circuit

Working of the circuit

(1) Input Circuit

In absence of input signal $V_i = 0$ as per the Kirchhoff's second law for input circuit

$V_{BB} = V_{BE}$

In presence of signal V_I, the change in the base emitter voltage is ΔV_{BE}

$\therefore V_{BB} + V_i = V_{BE} + \Delta V_{BE}$

From above equations

$V_i = \Delta V_{BE}$

Change in base current ΔI_B is due to the voltage change ΔV_{BE}. As per definition of the input resistance r_i we have

$$r_i = \frac{\Delta V_{BE}}{\Delta I_{BE}}$$

Or $\Delta V_{BE} = V_i = r_i \Delta I_B$

(2) Output circuit

The collector current increases by an amount ΔI_C due to the change in the base circuit ΔI_B. As a result the voltage change by an equal amount $R_L I_C$ across resistor R_L

As per the kirchhoff's law

$V_{CC} = I_C R_C + V_{CE}$

$\therefore \Delta V_{CC} = R_C \Delta I_C + \Delta V_{CE}$

As the battery V_{CC} remains constant $\Delta V_{CC} = 0$

Thus $\Delta V_{CE} = -R_C \Delta I_C$

Here ΔV_{CE} is obtained across the two ends of load resistor and is known as the output voltage V_O

$\therefore V_O = -R_C \Delta I_C$

Negative sign shows input and output voltages are out of phase by 180^O

Whenever the input voltage increases output voltage decrease and vice versa

Voltage gain(A_V):

As per the definition of voltage gain

$$A_V = \frac{output\ voltage}{Input\ voltage} = \frac{V_O}{V_i}$$

Since $V_O = -R_C \Delta I_C$ and $V_i = r_i \Delta I_B$

$$A_V = -\frac{R_C \Delta I_C}{r_i \Delta I_B}$$

As current gain $\beta = \Delta I_C / \Delta I_B$

$$A_V = -\beta \frac{R_C}{r_i}$$

Since trans-conductance $g_m = \beta / r_i$

$$\therefore A_V = -g_m R_C$$

Power gain (A_P): AS per definition of the gain A_P

$$A_P = \frac{Ouput\ AC\ power}{Input\ AC\ power}$$

$$A_P = \frac{\Delta V_{CE} \Delta I_C}{\Delta V_{BE} \Delta I_B}$$

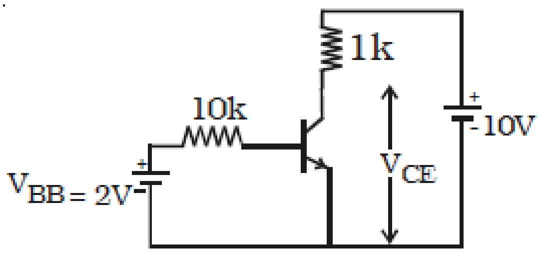
$$A_P = A_V A_i$$

$$A_P = (-\beta \frac{R_C}{r_i}) (\beta)$$

$$|A_P| = \beta^2 \frac{R_C}{r_i}$$

Solved Numerical

Q) The current gain β of the silicon transistor used in the circuit as shown in figure is 50. (Barrier potential for silicon is 0.69 V)



Find: (i) I_B (ii) I_E (iii) I_C and (iv) V_{CE}

Given:

$V_{BB} = 2$ V, $V_{CC} = 10$ V; β = 50 ;$R_B = 10$ k Ω ; $R_C = 1$ k Ω The barrier potential for silicon transistor $V_{BE} = 0.69$ V

Solution :

From input circuit

$V_{BB} = I_B R_B + V_{BE}$

$$I_B = \frac{V_{BB} - V_{BE}}{R_B} = \frac{2.0 - 0.69}{10000} = 131\mu A$$

Current gain β

$$\beta = \frac{I_C}{I_B}$$

$$I_C = \beta I_B$$

$$I_C = 50 \times 131 \times 10^{-6} = 6.5mA$$

Emitter current $I_E = I_C + I_B$

$I_E = 6.5$ mA + 131 μA

$I_E = 6.5$ mA + 0.131 mA

$I_E = 6.631$ mA

$V_{CC} = V_{CE} + I_C R_C$

$V_{CE} = V_{CC} - I_C R_C$

$V_{CE} = 10 - (6.5 \times 10^{-3} \times 1 \times 10^3)$

$V_{CE} = 3.5$ V

Q) A transistor is connected in CE configuration. The voltage drop across the load resistance (R_C) 3 k Ω is 6 V. Find the base current. The current gain α of the transistor is 0.97

Given : Voltage across the collector load resistance (R_C) = 6 V α = 0.97; R_C = 3 k Ω

Solution : The voltage across the

collector resistance is, $R_C = I_C R_C = 6V$

Hence, I_C

$$I_c = \frac{6}{R_C} = \frac{6}{3 \times 10^3} = 2\ mA$$

Current gain β

$$\beta = \frac{\alpha}{1-\alpha} = \frac{0.97}{1-0.97} = 32.33$$

$$I_B = \frac{I_C}{\beta} = \frac{2 \times 10^{-3}}{32.33} = 61.86\ \mu A$$

Q) A change of 0.02V takes place between the base and emitter when an input signal is connected to CE transistor amplifier. As a result, 20μA change takes place in the base current and change of 2mA takes place in the collector current. Calculate the following quantities

(1) Input resistance (2) A.C. Current gain (3) Trans-conductance (4) If the load resistance is 5kΩ, what will be the voltage gain and power gain

Solution:

Here $\Delta I_B = 20\mu A = 20\times10^{-6}$ A ; ΔV_{BE} = 0.02 V ; ΔI_C = 20 mA = 20×10^{-3} A, $R_L = 5\ k\Omega = 5\times10^3\ \Omega$

(1) Input resistance

$$r_i = \frac{\Delta V_{BE}}{\Delta I_B} = \frac{0.02}{20 \times 10^{-6}} = 1k\Omega$$

(2) AC current gain

$$A_i = \beta = \frac{\Delta I_C}{\Delta I_B} = \frac{2 \times 10^{-3}}{20 \times 10^{-6}} = 100$$

(3) Trans-conductance

$$g_m = \frac{\beta}{r_i} = \frac{100}{1000} = 0.1\ mho$$

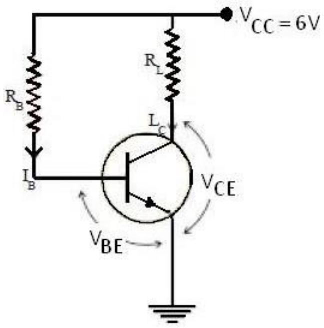
(4) Voltage gain

$$|A_V| = g_m R_L$$
$$|A_V| = (0.1)(5000) = 500$$

(5) Power gain

$$A_P = A_V A_i$$
$$A_P = (500)(100) = 5 \times 10^4$$

Q) For the circuit shown in figure I_B = 5μA. R_B = 1 MΩ., R_L = 1.1 kΩ, I_C = 5 mA and V_{CC} = 6V. Calculate the values of V_{BE} and V_{CE}



<u>Solution:</u>
For input circuit $V_{CC} = I_B R_B + V_{BE}$
$V_{BE} = V_{CC} - I_B R_B$
$V_{BE} = 6 – (5\times10^{-6})(1\times10^{6}) = 1$ V
From output circuit $V_{CC} = I_C R_C + V_{CE}$
$V_{CE} = V_{CC} - I_C R_C$
$V_{CE} = 6 – (5\times10^{-3})(1.1\times10^{3}) = 0.5$ V

Q) The A.C current gain of a PNP common emitter circuit is 100. The value of the input resistance is 1kΩ. What should be the value of the load resistor R_L in order to obtain power gain of 2000?
<u>Solution:</u>
Power gain

$$|A_P| = \beta^2 \frac{R_L}{r_i}$$

$$2000 = (100^2)\frac{R_L}{1\times10^3}$$

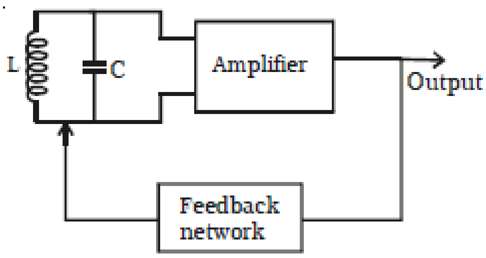
R_L = 200 Ω

Transistor oscillators

An oscillator may be defined as an electronic circuit which converts energy from a d.c. source into a periodically varying output.
Oscillators are classified according to the output voltage, into two types viz. sinusoidal and non-sinusoidal oscillators.
If the output voltage is a sine wave function of time, the oscillator is said to be sinusoidal oscillator. If the oscillator generates non-sinusoidal waveform, such as square, rectangular waves, then it is called as non-sinusoidal oscillator (multivibrator).
The oscillators can be classified according to the range of frequency as audio-frequency (AF) and radio-frequency (RF) oscillators.
Sinusoidal oscillators may be any one of the following three types:
(i) LC oscillators

(ii) RC oscillators
(iii) Crystal oscillators
Essentials of LC oscillator:
Fig shows the block diagram of an oscillator.

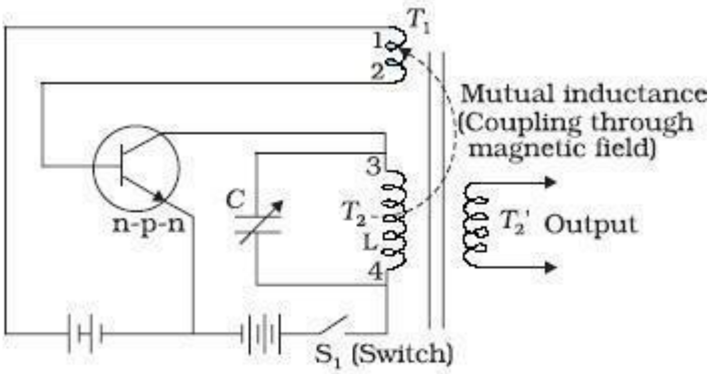
It's essential components are (i) tank circuit, ii) amplifier and (iii) feedback circuit.



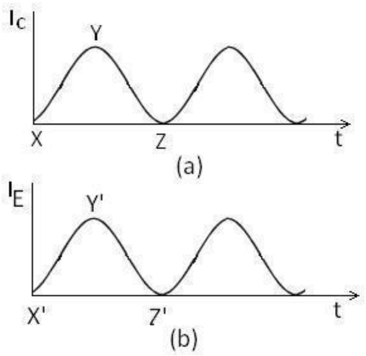
(i) Tank circuit : It consists of inductance coil (L) connected in parallel with capacitor (C). The frequency of oscillations in the circuit depends upon the values of inductance coil and capacitance of the capacitor.

(ii) Amplifier : The transistor amplifier receives d.c. power from the battery and changes it into a.c. power for supplying to the tank circuit.

(iii) Feedback circuit : It provides positive feedback (i.e.) this circuit transfers a part of output energy to LC circuit in proper phase, to maintain the oscillations



Suppose switch S_1 is put *on* to apply proper bias for the first time. Obviously, a *surge* of collector current flows in the transistor. This current flows through the coil T_2 where terminals are numbered 3 and 4 [Fig.]. This current does not reach full amplitude instantaneously but increases from X to Y, as shown in [Fig.a]
The inductive coupling between coil T_2 and coil T_1 now causes a current to flow in the emitter circuit (note that this actually is the 'feedback' from input to output).
As a result of this positive feedback, this current (in T_1; emitter current) also increases from X′ to Y′ [Fig.b]



The current in T_2 (collector current) connected in the collector circuit acquires the value Y when the transistor becomes *saturated*.
This means that maximum collector current is flowing and can increase no further. Since there is no further change in collector current, the magnetic field around T_2 ceases to grow. As soon as the field becomes static, there will be no further feedback from T_2 to T_1. Without continued feedback, the emitter current begins to fall.
Consequently, collector current decreases from Y towards Z [Fig. a]. However, a decrease of collector current causes the magnetic field to decay around the coil T_2. Thus, T_1 is now seeing a decaying field in T_2 (opposite from what it saw when the field was growing at the initial *start* operation). This causes a further decrease in the emitter current till it reaches Z' when the transistor is *cut-off*.
This means that both I_E and I_C cease to flow. Therefore, the transistor has reverted back to its original state (when the power was first switched on).
The whole process now repeats itself. That is, the transistor is driven to saturation, then to cut-off, and then back to saturation. The time for change from saturation to cut-off and back is determined by the constants of the tank circuit or tuned circuit (inductance *L* of coil T_2 and *C* connected in parallel to it). The resonance frequency (ν) of this tuned circuit determines the frequency at which the oscillator will oscillate.

$$\nu = \frac{1}{2\pi\sqrt{LC}}$$

In the circuit of the tank or tuned circuit is connected in the collector side. Hence, it is known as *tuned collector oscillator*.
If the tuned circuit is on the base side, it will be known as *tuned base oscillator*.
There are many other types of tank circuits (say *RC*) or feedback circuits giving different types of giving different types of oscillators like Colpitt's oscillator, Hartley oscillator, *RC*-oscillator

Solved Numerical

Q) In transistor oscillator circuit an output signal of 1MHz frequency is obtained. The value of capacitance C = 100pF. What should be the value of the capacitor is a signal of 2MHz frequency is to be obtained
Solution:
C_1 = 100pF = 10×10^{-13} F, f_1 = 1MHz = 10^6 Hz

f_2 = 2 MHz = 2×10^6 Hz, C_2 = ?

$$f_1 = \frac{1}{2\pi\sqrt{LC_1}}$$

And

$$f_2 = \frac{1}{2\pi\sqrt{LC_2}}$$

$$\frac{f_1}{f_2} = \sqrt{\frac{C_2}{C_1}}$$

$$C_2 = \left(\frac{f_1}{f_2}\right)^2 \times C_1$$

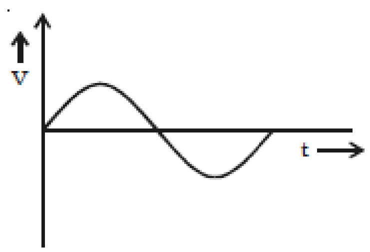
$$C_2 = \left(\frac{1}{2}\right)^2 \times 100 \times 10^{-12} = 25pF$$

Digital electronics

Digital Electronics involves circuits and systems in which there are only two possible states which are represented by voltage levels. Other circuit conditions such as current levels open or closed switches can also represent the two states.

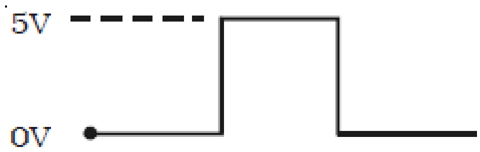
Analog signal

The signal current or voltage is in the form of continuous, time varying voltage or current (sinusoidal). Such signals are called continuous or analog signals. A typical analog signal is shown in Fig



Digital signal and logic levels

A digital signal (pulse) is shown in Fig. It has two discrete levels, 'High' and 'Low'. In most cases, the more positive of the two levels is called HIGH and is also referred to as logic 1. The other level becomes low and also called logic 0. This method of using more positive voltage level as logic 1 is called a positive logic system. A voltage 5V refers to logic 1 and 0 V refers to logic 0. On the other hand, in a negative logic system, the more negative of the two discrete levels is taken as logic 1 and the other level as logic 0. Both positive and negative logic are used in digital systems. But, positive logic is more common of logic gates. Hence we consider only positive logic for studying the operation of logic gates.



Logic gates

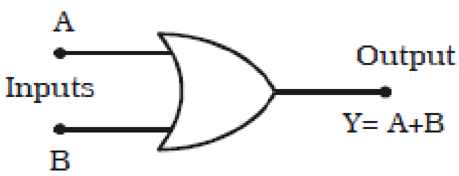
Circuits which are used to process digital signals are called logic gates. They are binary in nature. Gate is a digital circuit with one or more inputs but with only one output. The output appears only for certain combination of input logic levels. Logic gates are the basic building blocks from which most of the digital systems are built up. The numbers 0 and 1 represent the two possible states of a logic circuit. The two states can also be referred to as 'ON and OFF' or 'HIGH and LOW' or 'TRUE and FALSE'.

Basic logic gates using discrete components

The basic elements that make up a digital system are 'OR', 'AND' and 'NOT' gates. These three gates are called basic logic gates. All the possible inputs and outputs of a logic circuit are represented in a table called TRUTH TABLE. The function of the basic gates are explained below with circuits and truth tables.

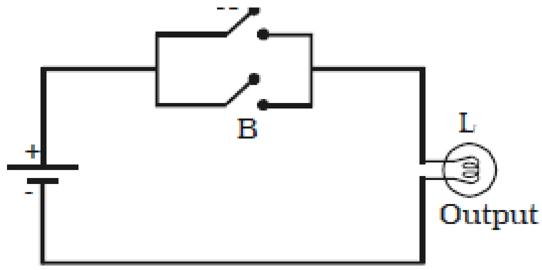
(i) OR gate

An OR gate has two or more inputs but only one output. It is known as OR gate, because the output is high if any one or all of the inputs are high. The logic symbol of a two input OR gate is shown in Fig

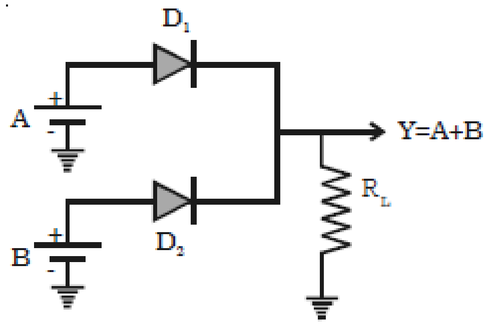


The Boolean expression to represent OR gate is given by Y= A+B (+ symbol should be read as OR)

The OR gate can be thought of like an electrical circuit shown in Fig, in which switches are connected in parallel with each other. The lamp will glow if both the inputs are closed or any one of them is closed.



Diode OR gate
Fig shows a simple circuit using diodes to build a two input OR gate. The working of this circuit can be explained as follows.

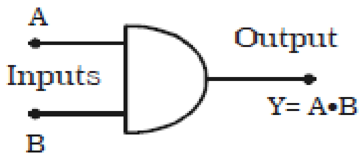


Case (i) A = 0 and B = 0
When both A and B are at zero level, (i.e.) low, the output voltage will be low, because the diodes are non-conducting.
Case (ii) A = 0 and B = 1
When A is low and B is high, diode D_2 is forward biased so that current flows through RL and output is high.
Case (iii) A = 1 and B = 0
When A is high and B is low, diode D_1 conducts and the output is high.
Case (iv) A = 1 and B = 1
When A and B both are high, both diodes D_1 and D_2 are conducting and the output is high. Therefore Y is high
Truth table of OR gate:

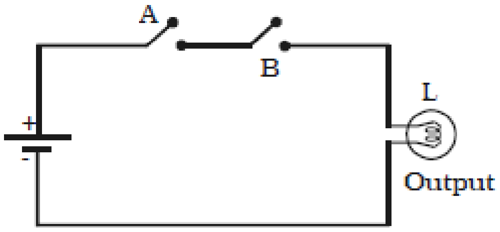
Inputs		Output
A	B	Y = A + B
0	0	0
0	1	1
1	0	1
1	1	1

(ii) AND gate

An AND gate has two or more inputs but only one output. It is known as AND gate because the output is high only when all the inputs are high. The logic symbol of a two input AND gate is shown in Fig.



The Boolean expression to represent AND gate is given by Y = A·B (· should be read as AND) AND gate may be thought of an electrical circuit as shown in Fig,



in which the switches are connected in series. Only if A and B are closed, the lamp will glow, and the output is high.

Diode AND gate

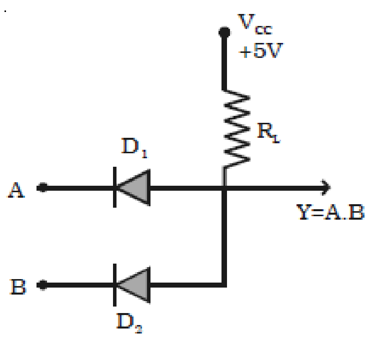


Fig shows a simple circuit using diodes to build a two-input AND gate.

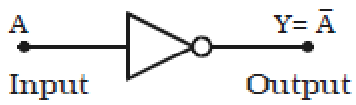
The working of the circuit can be explained as follows :
Case (i) A = 0 and B = 0
When A and B are zero, both diodes are in forward bias condition and they conduct and hence the output will be zero, because the supply voltage V_{CC} will be dropped across RL only. Therefore Y = 0.
Case (ii) A = 0 and B = 1
When A = 0 and B is high, diode D_1 is forward biased and diode D_2 is reverse biased. The diode D_1 will now conduct due to forward biasing. Therefore, output Y = 0.
Case (iii) A = 1 and B = 0
In this case, diode D_2 will be conducting and hence the output Y = 0.
Case (iv) A = 1 and B = 1
In this case, both the diodes are not conducting. Since D1 and D2 are in OFF condition, no current flows through RL. The output is equal to the supply voltage. Therefore Y = 1.
Thus the output will be high only when the inputs A and B are high.
Truth table of AND gate:

Inputs		Output
A	B	$Y = A \cdot B$
0	0	0
0	1	0
1	0	0
1	1	1

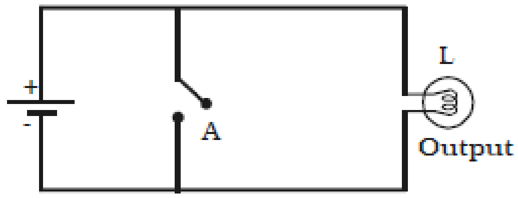
.

(iii) NOT gate (Inverter)

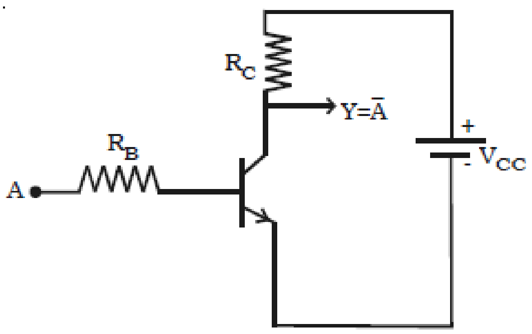
The NOT gate is a gate with only one input and one output. It is so called, because its output is complement to the input. It is also known as inverter. Fig shows the logic symbol for NOT gate.



The Boolean expression to represent NOT operation is Y = *A* .
The NOT gate can be thought of like an electrical circuit as shown in Fig.



When switch A is closed, input is high and the bulb will not glow (i.e) the output is low and vice versa.
When the input A is high, the transistor is driven into saturation and hence the output Y is low. If A is low, the transistor is in cutoff and hence the output Y is high. Hence, it is seen that whenever input is high, the output is low and vice versa.

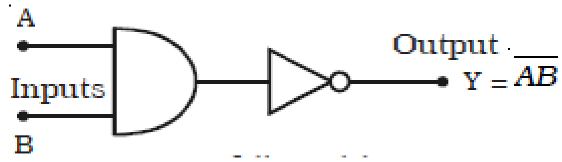


Input	Output
A	$Y = \bar{A}$
0	1
1	0

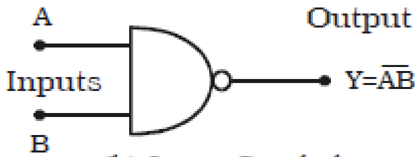
Truth table of NOT gate:

(iv) NAND gate

This is a NOT–AND gate. It can be obtained by connecting a NOT gate at the output of an AND gate (Fig).



The logic symbol for NAND gate is shown in Fig 9.53b.



(b) Logic Symbol

The Boolean expression to represent NAND Operation is Y = *AB*

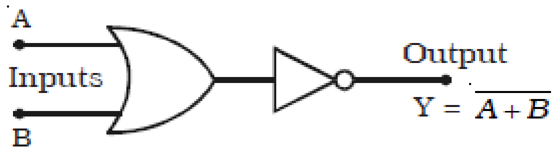
NAND gate function is reverse of AND gate function. A NAND gate will have an output, only if both inputs are not 1. In other words, it gives an output 1, if either A or B or both are 0.

Truth table of NAND gate:

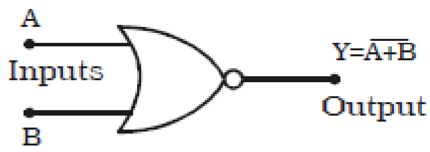
Inputs		Output Y = AB
A	B	
0	0	1
0	1	1
1	0	1
1	1	0

(V)NOR gate

This is a NOT–OR gate. It can be made out of an OR gate by connecting an inverter at its output (Fig).



The logic symbol for NOR gate is given in Fig.



The Boolean expression to represent NOR gate is Y = $\overline{A+B}$

The NOR gate function is the reverse of OR gate function. A NOR gate will have an output, only when all inputs are 0. In a NOR gate, output is high, only when all inputs are low.

Truth table of NOR gate:

Inputs		Output
A	B	$Y = \overline{A+B}$
0	0	1
0	1	0
1	0	0
1	1	0

De-Morgan's theorems

First theorem

"The complement of a sum is equal to the product of the complements."

If A and B are the inputs, then

$$\overline{A+B} = \overline{A}\cdot\overline{B}$$

Second theorem

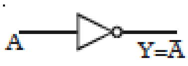
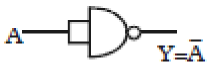
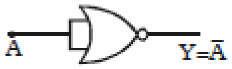
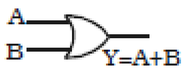
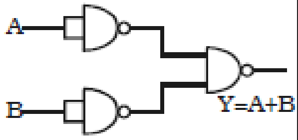
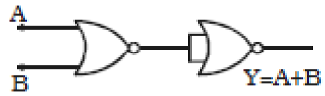
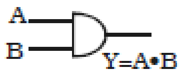
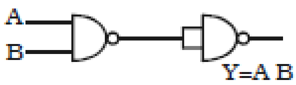
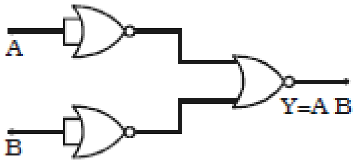
"The complement of a product is equal to the sum of the complements." If A and B are the inputs, then $\overline{AB} = \overline{A} + \overline{B}$

The theorems can be proved, first by considering the two variable cases and then extending this result as shown in Table

A	B	$\overline{A}$	$\overline{B}$	$\overline{A\,.\,B}$	$\overline{A}+\overline{B}$	$\overline{A+B}$	$\overline{A}\,.\,\overline{B}$
0	0	1	1	1	1	1	1
0	1	1	0	1	1	0	0
1	0	0	1	1	1	0	0
1	1	0	0	0	0	0	0

NAND and NOR as Universal gates

NAND and NOR gates are called Universal gates because they can perform all the three basic logic functions. Table gives the construction of basic logic gates NOT, OR and AND using NAND and NOR gates

Logic function	Symbol	Circuits using NAND gates only	Circuits using NOR gates only
NOT			
OR			
AND			

Boolean algebra

Boolean algebra, named after a mathematician George Boole is the algebra of logic, which is applied to the operation of computer devices. The rules of this algebra is simple, speed and accurate. This algebra is helpful in simplifying the complicated logical expression.

Laws and theorems of Boolean algebra

The fundamental laws of Boolean algebra are given below which are necessary for manipulating different Boolean expressions.

Basic laws :

Commutative laws : A + B = B + A ; AB = BA

Associative Laws: A + (B + C) = (A + B) + C ; A (BC) = (AB) C

Distributive law: A (B+C) = AB + AC

Special theorems :

A + AB = A

(A + B) (A + C) = A + BC

A (A + B) = A

A + *A* B = A + B

A (*A* + B) = AB

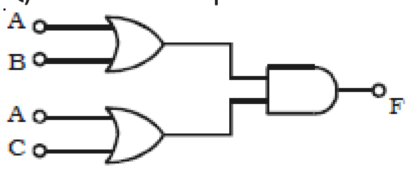
(A + B) (*A* + C) = AC + *A* B

AB + *A* C = (A + C) (*A* + B)

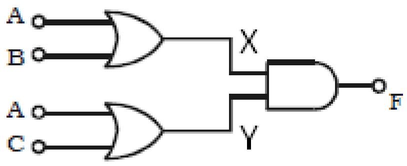
Theorems involving a single variable can be proved by considering every possible value of the variable.

Solved Numerical

Q) Find the output F of the logic circuit given below:



Solution :



Let X and Y be the output of two OR gates

Thus X = A + B

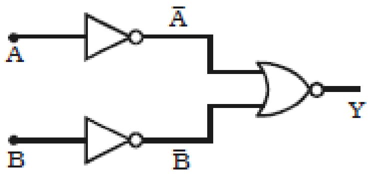
And Y = A + C

Output X and Y acts as input for AND gate

Thus F = X · Y

F = (A+B)(A+C) = AC + AB (from special theorems)

Q) The outputs of two NOT gates are in put for NOR, as shown in figure. What is this combination equivalent to?



Solution:

From the logic circuit it follows that the output

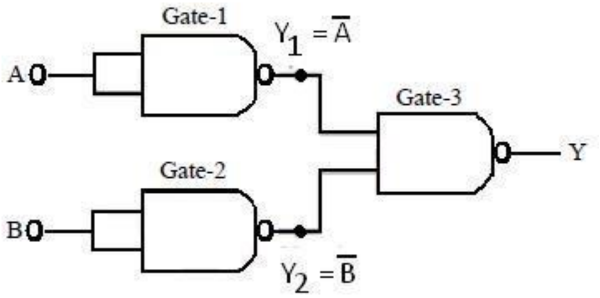
$$y = \overline{\bar{A} + \bar{B}}$$

Applying DeMorgan's first theorem,

we get, $y = \bar{\bar{A}} \cdot \bar{\bar{B}} = AB$

Hence given logic circuit is AND operation.

Q) Show that the circuit drawn in figure comprising of three NAND gates behave as an OR gate



Solution:
Gate 1 and Gate 2 have identical inputs. Hence both behaves as NOT gate
Hence $y_1 = \bar{A}$ and $y_2 = B$
Gate 3 is NAND hence output

$$y = \overline{y_1}\overline{y_2}$$

$$y = \overline{\bar{A}\bar{B}}$$

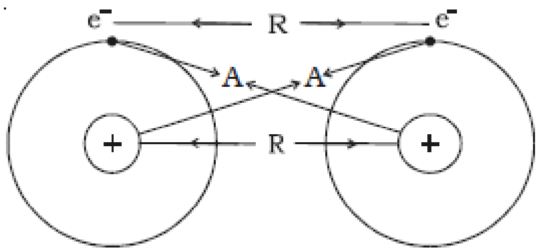
By DeMorgan's theorem y = AB
Hence the above circuit behaves as OR gate and can be verified using truth table

SOLIDS AND FLUIDS

ELASTICITY

In solids, the atoms and molecules are free to vibrate about their mean positions. If this vibration increases sufficiently, molecules will shake apart and start vibrating in random directions. At this stage, the shape of the material is no longer fixed, but takes the shape of its container. This is liquid state. Due to increase in their energy, if the molecules vibrate at even greater rates, they may break away from one another and assume gaseous state. Water is the best example for this changing of states. Ice is the solid form of water. With increase in temperature, ice melts into water due to increase in molecular vibration.
If water is heated, a stage is reached where continued molecular vibration results in a separation among the water molecules and therefore steam is produced. Further continued heating causes the molecules to break into atoms.

Intermolecular or inter atomic forces
Consider two isolated hydrogen atoms moving towards each other as shown in Fig As they approach each other, the following interactions are observed.



(i) Attractive force A between the nucleus of one atom and electron of the other. This attractive force tends to decrease the potential energy of the atomic system.
(ii) Repulsive force R between the nucleus of one atom and the nucleus of the other atom and electron of one atom with the electron of the other atom. These repulsive forces always tend to increase the energy of the atomic system. There is a universal tendency of all systems to acquire a state of minimum potential energy. This stage of minimum potential energy corresponds to maximum stability. If the net effect of the forces of attraction and repulsion leads to decrease in the energy of the system, the two atoms come closer to each other and form a covalent bond by sharing of electrons. On the other hand, if the repulsive forces are more and there is increase in the energy of the system, the atoms will repel each other and do not form a bond. The forces acting between the atoms due to electrostatic interaction between the charges of the atoms are called inter atomic forces. Thus, inter atomic forces are electrical in nature. The inter atomic forces are active if the distance between the two atoms is of the order of atomic size ≈ 10^{-10} m. In the case of molecules, the range of the force is of the order of 10^{-9} m.

Elasticity
When an external force is applied on a body, which is not free to move, there will be a relative displacement of the particles. Due to the property of elasticity, the particles tend to regain their original position. The external forces may produce change in length, volume and shape of the body.

This external force which produces these changes in the body is called deforming force. A body which experiences such a force is called deformed body.

When the deforming force is removed, the body regains its original state due to the force developed within the body. This force is called restoring force.

The property of a material to regain its original state when the deforming force is removed is called elasticity.

The bodies which possess this property are called elastic bodies. Bodies which do not exhibit the property of elasticity are called plastic. The study of mechanical properties helps us to select the material for specific purposes. For example, springs are made of steel because steel is highly elastic

Stress and strain

In a deformed body, restoring force is set up within the body which tends to bring the body back to the normal position. The magnitude of these restoring force depends upon the deformation caused. *This restoring force per unit area of a deformed body is known as stress.* This is measured by the magnitude of the deforming force acting per unit area of the body when equilibrium is established.

$$Stree = \frac{restoring\ force}{Area}$$

Unit of stress in S.I. system is N/m^2. When the stress is normal to the surface, it is called Normal Stress. The normal stress produces a achange in length or a change in volume of the body. The normal stress to a wire or a body may be compressive or tensile (expansive) according as it produces a decrease or increase in length of a wire or volume of the body. When the stress is tangential to the surface, it is called tangential (shearing) stress

Solved Numerical

Q) A rectangular bar having a cross-sectional area of 28 mm^2 has a tensile force of a 7KN applied to it. Determine the stress in the bar

Solution

Cross-sectional area A = 25mm^2 = 28×(10^{-3})^2= 28× 10^{-6} m^2

Tensile force F = 7KN = 7×10^{3}N

$$Stree = \frac{7 \times 10^3}{28 \times 10^{-6}} = 0.25 \times 10^9\ N/m^2$$

Strain

The external force acting on a body cause a relative displacement of its various parts. A change in length volume or shape takes place. The body is then said to be strained. The relative change produced in the body under a system of force is called strain

$$Strain\ (\varepsilon) = \frac{Change\ in\ dimension}{original\ dimension}$$

Strain has no dimensions as it is a pure number. The change in length per unit length is called linear strain. The change in volume per unit volume is called Volume stain. If there is a change in shape the strain is called shearing strain. This is measured by the angle through which a line originally normal to the fixed surface is turned

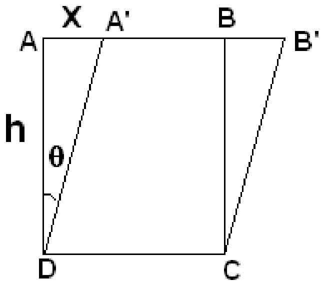
Longitudinal Strain: The ratio of change in length to original length

$$\varepsilon_l = \frac{\Delta l}{l}$$

Volume strain

$$\varepsilon_v = \frac{\Delta v}{v}$$

Shearing strain

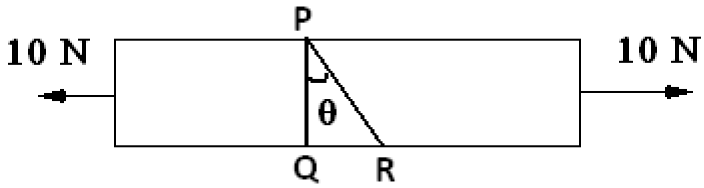


In figure a body with square cross section is shown a tangential force acts on the top surface AB causes shift of Surface by 'X' units shown as surface A'B' ,thus side DA' now mates an angle of θ with original side DA of height h

$$\varepsilon_S = \frac{x}{h} = tan\theta$$

Solved Numerical

Q) As shown in figure 10N force is applied at two ends of a rod. Calculate tensile stress and shearing stress for section PR. Area of cross-section PQ is 10 cm^2 , θ=30^O



Solution

Given cross-section area of PQ = 10 cm^2

Now PQ = PRcosθ

10 = PRcos30

10= PR (√3 /2)

PR = 20/√3 cm^2 or 2/√3 m^2

Now normal force to area PR will be Fcos30 = 10 ×(√3 /2) = 5√3 N

Tangential force to area PR will be Fsin30 = 10×(1/2) = 5 N

∴ Tensile stress for section PR

$$\sigma_l = \frac{normal\ force}{area\ of\ PR} = \frac{5\sqrt{3}}{\frac{2}{\sqrt{3}} \times 10^{-3}} = 7.5 \times 10^3 \frac{N}{m^2}$$

Shearing stress for section PR

$$\sigma_t = \frac{tangential\ force}{area\ of\ PR} = \frac{5}{\frac{2}{\sqrt{3}} \times 10^{-3}} = 2.5\sqrt{3} \times 10^3 \frac{N}{m^2}$$

Hooke's Law and types of moduli

According to Hooke's law, *within the elastic limit, strain produced in a body is directly proportional to the stress that produces it.*

$$\frac{stress}{strain} = constant = \lambda$$

Where λ is called modulus of elasticity.

Its unit is N m^{-2} and its dimensional formula is $ML^{-1}T^{-2}$.

Depending upon different types of strain, the following three moduli of elasticity are possible

(i) Young's modulus: When a wire or rod is stretched by a longitudinal force the ratio of the longitudinal stress to the longitudinal strain within the elastic limits is called Young's modulus

$$Young's modulus\ (Y) = \frac{Longitudinal\ stress}{linear\ strain}$$

Consider a wire or rod of length L and radius r under the action of a stretching force applied normal to its face. Suppose the wire suffers a change in length l then

$$Longitudinal\ stress = \frac{F}{\pi r^2}$$

$$Linear\ strain = \frac{l}{L}$$

$$Young's modulus\ (Y) = \frac{\frac{F}{\pi r^2}}{\frac{l}{L}} = \frac{FL}{\pi r^2 l}$$

(ii) Bulk modulus: When a solid or fluid (liquid or gas) is subjected to a uniform pressure all over the surface, the shape remains the same, but there is a change in volume. The force perunit area applied normally and uniformly over the surface is called normal stress. The change in volume per unit volume is called volume or bulk strain.

$$Bulk\ modulus\ (B) = \frac{Volume\ \ stress}{Volume\ strain}$$

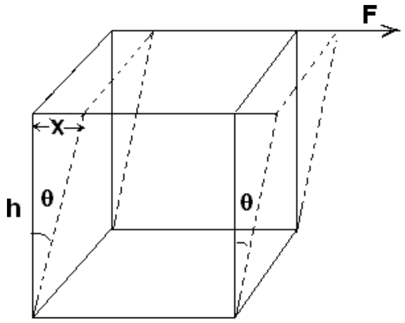
$$B = \frac{-\frac{F}{A}}{\frac{\Delta V}{V}} = -\frac{FV}{A\Delta V}$$

Negative sign indicate reduction in volume

The reciprocal of bulk modulus is called compressibility

$$compressibility = \frac{1}{bulk\ modulus}$$

(iii) Modulus of rigidity: According to the definition, the ratio of shearing stress to shearing strain is called modulus of rigidity (η) . in this case the sphape of the body changes but its volume remains unchanged. Consider the case of a cube fixed at its lower face and acted upon by a tangential force F on its upper surface of area A as shown in figure



$$shearing\ stress = \frac{F}{A}$$

$$Shearing\ strain = \theta = \frac{x}{h}$$

$$\eta = \frac{F}{A\theta} = \frac{Fh}{Ax}$$

Solved Numerical

Q) A solid sphere of radius R made of a material of bulk modulus B is surrounded by a liquid in cylindrical container. A massless piston of area A flots on the surface of the liquid. Find the fractional change in the radius of the sphere (dR/R) when a mass M is placed on the piston to compress the liquid

Solution

From the formula of Bulk modulus

$$B = -\frac{FV}{A\Delta V}$$

$$V = \frac{4}{3}\pi R^3$$

$$dV = 4\pi R^2 dR$$

$$B = -\frac{F\frac{4}{3}\pi R^3}{A4\pi R^2 dR}$$

$$\frac{dR}{R} = \frac{Mg}{3AB}$$

Q) Find the natural length of rod if its length is L_1 under tension T_1 and L_2 under tension T_2 within the limits of elasticity

Solution

From the formula of Young's modulus

$$Young's\ modulus\ (Y) = \frac{\frac{F}{A}}{\frac{l}{L}}$$

Let increase in length for tension T_1 be x and that for tension T_2 be y then

$$\frac{\frac{T_1}{A}}{\frac{x}{L}} = \frac{\frac{T_2}{A}}{\frac{y}{L}}$$

$$\frac{T_1}{x} = \frac{T_2}{y}$$

$$T_1 y = T_2 x$$

But x = L_1 – L and y = L_2 – L

$$T_1(L_2 - L) = T_2(L_1 - L)$$

On simplification we get

$$L = \frac{(L_1T_2 - L_2T_1)}{(T_2 - T_1)}$$

Q) A copper wire of negligible mass, 1 m length and cross-sectional area $10^{-6}m^2$ is kept on a smooth horizontal table with one end fixed. A ball of mass 1kg is attached to the other end. The wire and the ball are rotated with an angular velocity of 20 rad/s. if the elongation in the wire is 10^{-3}m, obtain the Young's modulus. If on increasing the angular velocity to 100 rad/s the wire breaks down, obtain the breaking stress.

Solution

Given m = 1kg, ω= 20 rad/s, L = 1m ΔL = 10^{-3} m, A = 10^{-6} m^2

Tension in the thread

T = $m\omega^2 L = 1\times(20)^2\times1$ = 400N

$$Y = \frac{TL}{A\Delta L} = \frac{400 \times 1}{10^{-6} \times 10^{-3}} = 4 \times 10^{11} N/m^2$$

On increasing the angular velocity to 100 rad/s, the wire breaks down then

$$breaking\ stress = \frac{T'}{A} = \frac{m(\omega')^2 L}{A}$$

$$breaking\ stress = \frac{1 \times (100)^2 \times 1}{10^{-6}} = 10^{10}\ N/m^2$$

Q) A cube is subjected to pressure of 5×10^5 N/m^2. Each side of the cubic is shorteed by 1%. Find volumetric strain and bulk modulus of elasticity of cube

Solution

V = l^3

Now dV = $3l^2$ dl

Thus

$$\frac{dV}{V} = \frac{3l^2 dl}{l^3} = 3\frac{dl}{l}$$

Sides are reduced by 1% thus dl/l =- 0.01

Thus reduction in volume = -0.03

Normal stress = Increase in pressure

$$B = -\frac{P}{\frac{\Delta V}{V}} = \frac{5 \times 10^5}{0.03} = 1.67 \times 10^7 N/m^2$$

Q) A rubber cube of each side 7cm has one side fixed, while a tangential force equal to the weight of 300kg f is applied to the opposite face. Find the shearing strain produced and the distance through which the strained site moves. The modulus of rigidity for rubber is 2×10^7 dyne/cm^2 g = 10m/s^2

Solution

Here L = 7cm = 7×10^{-2} m

F = 300 kg f = 300×10 N

Modulus of rigidity η = 2×10^7 dynes/cm^2 = 2×10^6 N/m^2

As

$$\eta = \frac{F}{A\theta}$$

$$\theta = \frac{F}{A\eta} = \frac{F}{h^2\eta}$$

$$\theta = \frac{3000}{(7\times10^{-2})^2\times 2\times 10^6} = 0.3\ rad$$

$$\theta = \frac{x}{h}$$

X = hθ

X = 7×0.3 = 2.1 cm

Poisson's Ratio:

It is the ratio of lateral strain to the longitudinal strain. For example, consider a force F applied along the length of the wire which elongates the wire along the length and it contracts radially. Then the longitudinal strain =Δl/l and lateral strain = Δr/r , where r is the radius of the wire

$$Poisson's\ ratio\ (\sigma) = -\frac{\frac{\Delta r}{r}}{\frac{\Delta l}{l}}$$

$$\frac{\Delta r}{r} = -\sigma\frac{\Delta l}{l}$$

For rectangular bar: let b be breadth and h be thickness then

$$\frac{\Delta b}{b} = -\sigma\frac{\Delta l}{l}$$

$$\frac{\Delta h}{h} = -\sigma\frac{\Delta l}{l}$$

The negative sign indicates that change in length and radius is of opposite sign. Change in volume due to longitudinal force

Due to application of tensile force, lateral dimension decreases and length increases. As a result there is a change in volume (usually volume increases). Let us consider the case of a cylindrical rod of length l and radius r.

Since $V=\pi r^2 L$

$$\therefore \frac{\Delta V}{V} = 2\frac{\Delta r}{r} + \frac{\Delta}{l} \quad (for\ very\ small\ change)$$

From above equations or radius and Length

$$\therefore \frac{\Delta V}{V} = -2\sigma\frac{\Delta l}{l} + \frac{\Delta l}{l}$$

$$\therefore \frac{\Delta V}{V} = \frac{\Delta l}{l}(1-2\sigma)$$

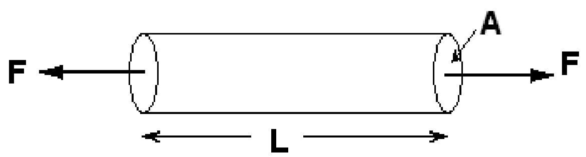
Longitudinal Strain: $\varepsilon_l = \frac{\Delta l}{l}$

$$\therefore \frac{\Delta V}{V} = \varepsilon_l(1-2\sigma)$$

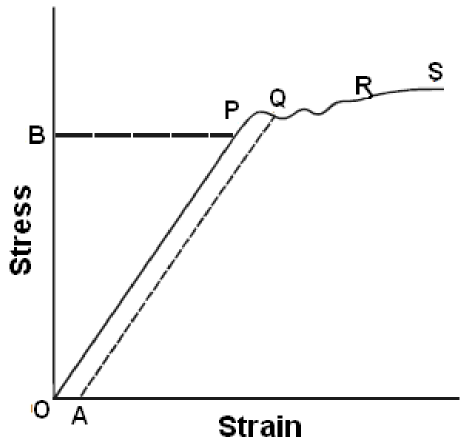
Above equation suggest that since $\Delta v > 0$, value of σ cannot exceed 0.5

Stress –Strain relationship for a wire subjected to longitudinal stress

Consider a long wire (made of steel) of cross-sectional area A and original length L in equilibrium under the action of two equal and opposite variable force F as shown in figure. Due to the application of force, the length gets changed to L +l. Then, longitudinal stress = F/A and Longitudinal strain = l/L



The extension of the wire is suitably measured and a stress – strain graph is plotted



(i) In the figure the region OP is linear. Within a normal stress, strain is proportional to the applied stress. This is Hooke's law. Up to P, when the load is removed the wire regains its original length along PO. The point P represents the **elastic limit**, PO represents the elastic range of the material and OB is the **elastic strength**.

(ii) Beyond P, the graph is not linear. In the region PQ the material is partly elastic and partly plastic. From Q, if we start decreasing the load, the graph does not come to O via P, but traces a straight line QA.

Thus a permanent strain OA is caused in the wire. This is called **permanent set**.

(iii) Beyond Q addition of even a very small load causes enormous strain. This point Q is called the yield point. The region QR is the **plastic range**.

(iv) Beyond R, the wire loses its shape and becomes thinner and thinner in diameter and ultimately breaks, say at S. Therefore S is the **breaking point**. The stress corresponding to S is called **breaking stress**.

Elastic potential energy or Elastic energy stored in a deformed body

The elastic energy is measured in terms of work done in straining the body within its elastic limit

Let F be the force applied across the cross-section A of a wire of length L. Let l be the increase in length. Then

$$Y = \frac{\frac{F}{A}}{\frac{l}{L}} = \frac{FL}{Al}$$

$$F = \frac{YAl}{L}$$

If the wire is stretched further through a distance of dl, the work done dw

$$dW = F \times dl = \frac{YAl}{L} dl$$

Total work done in stretching the wire from original length L to a length L +l (i.e. from l = 0 to l = l)

$$W = \int_0^l \frac{YAl}{L} dl$$

$$W = \frac{YA}{L}\frac{l^2}{2} = \frac{1}{2}(AL)\left(\frac{Yl}{L}\right)\left(\frac{l}{L}\right)$$

$$W = \frac{1}{2} \times volume \times stress \times strain$$

Solved Numerical

Q) The rubber cord of catapult has a cross-section area 1mm^2 and total unstrtched length 10 cm. It is stretched to 12cm and then released to project a body of mass 5g. taking the Young's modulus of rubber as 5×10^8 N/m^2, calculate the velocity of projection

Solution

It can be assumed that the total elastic energy of catapult is converted into kinetic energy of the body without any heat loss

L = 12cm = 12×10^{-2} m , l = 2cm = 2×10^{-3} m , A = 1mm^2 = 10^{-6}m

$$U = \frac{YA}{L}\frac{l^2}{2} = \frac{5 \times 10^8 \times (1 \times 10^{-6}) \times (2 \times 10^{-2})^2}{2 \times 10 \times 10^{-2}} = 1$$

Now K.E of projectile = elastic energy of catapult

$$\frac{1}{2} mv^2 = U$$

$$\frac{1}{2} \times 5 \times 10^{-3} \times v^2 = 1$$

V = 20 m/s

FLUID STATICS

Thrust and Pressure

A perfect fluid resists force normal to its surface and offers no resistance to force acting tangential to it surface. A heavy log of wood can be drawn along the surface of water with very little effort because the force applied on the log of wood is horizontal and parallel to the surface of water. Thus fluids are capable of exerting normal stress on the surface with it is in contact

Force exerted perpendicular to a surface is called thrust and thrust per unit area is called pressure

Variation of pressure with height

Let *h* be the height of the liquid column in a cylinder of cross sectional area *A*. If ρ is the density of the liquid, then weight of the liquid column *W* is given by

W = mass of liquid column × $g = Ah\rho g$

By definition, pressure is the force acting per unit area.

$$Pressure = \frac{weight\ of\ liquid\ column}{area\ of\ cross - \text{section}}$$

$$P = \frac{Ah\rho g}{A} = h\rho g$$

$dP = \rho g\ (dh)$

This differential relation shows that the pressure in a fluid increases with depth or decreases with increased elevation. Above equation holds for both liquids and gases. Liquids are generally treated as incompressible and we may consider their density ρ constant for every part of liquid. With ρ as constant, equation may be integrated as it stands, and the result is

$P = P_0 + \rho gh$

The pressure P_0 is the pressure at the surface of the liquid where h = 0

Force due to fluid on a plane submerged surface

The pressure at different points on the submerged surface varies so to calculate the resultant force, we divide the surface into a number of elementary areas and we calculate the force on it first by treating pressure as constant then we integrate it to get the net force i.e $F_R = \int P\ (dA)$

The point of application of resultant force must be such that the moment of the resultant force about any axis is equal to the moment of the distributed force about the axis

Solved Numerical

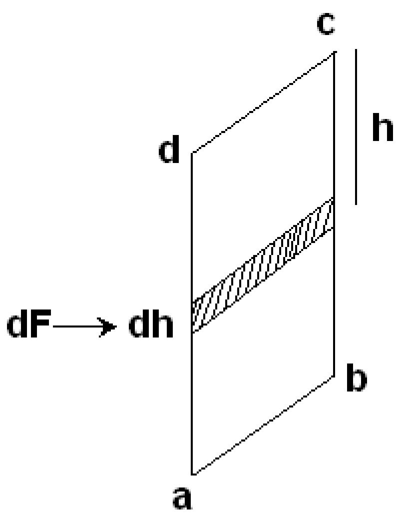
Q) Water is filled upto the top in a rectangular tank of square cross-section. The sides of cross-section is a and height of the tank is H. If density of water is ρ, find force on the bottom of the tank and on one of its wall. Also calculate the position of the point of application of the force on the wall

Solution
Force on the bottom of thank
Area of bottom of tank = a^2
Force = pressure × Area
Force = $H\rho g a^2$
Force on the wall and its point of application
Force on the wall of the tank is different at different heights so consider a segment at depth h of thickness dh
Pressure at depth h = $h\rho g$
Area of strip = a dh
Force on strip dF = $h\rho g$ a dh
Total force at on the wall

$$F = \int_0^H \rho g a h \, dh$$

$$F = \rho g a \left[\frac{h^2}{2}\right]_0^H$$

$$F = \rho g a \frac{H^2}{2}$$



The point of application of the force on the wall can be calculated by equating the moment of resultant force about any line, say dc to the moment of distributioed force about the same line dc
Moment of dF about line cd = dF (h) = ($h\rho g a dh$) h = $\rho g a h^2$ dh
∴ Net moment of distributed forces

$$\rho g a \int_0^H h^2 dh = \rho g a \frac{H^3}{3}$$

Let the point of application of the net force is at a depth 'x' from the line cd
Then the torque of the resultant force about the line cd =
$Fx = \rho g a \frac{H^2}{2} x$
Now Net moment of distribution of force = Torque

$$\rho g a \frac{H^3}{3} = \rho g a \frac{H^2}{2} x$$

$$x = \frac{2H}{3}$$

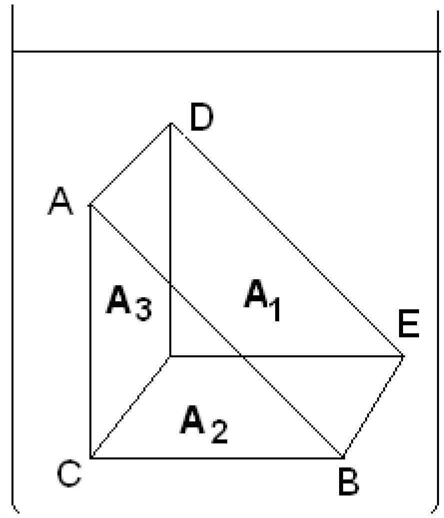
Hence, the resultant force on the vertical wall of the tank will act at a depth 2H/3 from the free surface of water or at the height of H/3 from bottom of tank

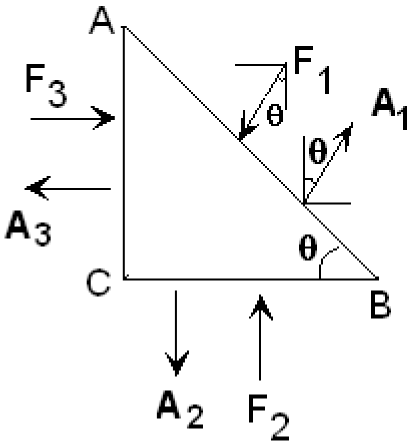
Pascal’s Law

Pascal’s law states that if the effect of gravity can be neglected then the pressure in an incompressible fluid in equilibrium is the same everywhere..

This statement can be verified as follows

Consider a small element of liquid in the interior of the liquid at rest. The liquid element is in the shape of prism consisting of two right angled triangle surfaces





Let the areas of surface ADEB, CFEB, ADFC be A_1, A_2, A_3

It is clear from figure that

$A_2 = A_1\cos\theta$ and $A_3 = A_1\sin\theta$

Also, since liquid element is in equilibrium $F_3 = F_1\cos\theta$ and $F_3 = F_1\sin\theta$

now pressure on surface ADEB is $P_1 = F_1 / A_1$

Pressure on the surface CFED is

$$P_2 = \frac{F_2}{A_2} = \frac{F_1 cos\theta}{A_1 cos\theta} = \frac{F_1}{A_1}$$

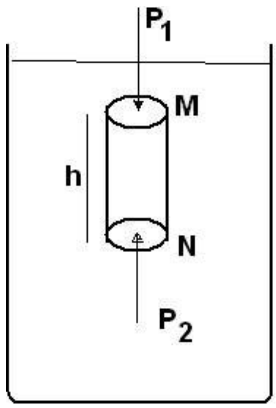
And pressure on the surface ADFC is

$$P_3 = \frac{F_3}{A_3} = \frac{F_1 sin\theta}{A_1 sin\theta} = \frac{F_1}{A_1}$$

So, $P_1 = P_2 = P_3$

Since θ is arbitrary this result holds for any surface. Thus Pascal’s law is verified

Pascal’s law and effect of gravity



When gravity is taken into account, Pascal’s law is to be modified. Consider a cylindrical liquid column of height *h* and density ρ in a vessel as shown in the Fig.

If the effect of gravity is neglected, then pressure at *M* will be equal to pressure at *N*.

But, if force due to gravity is taken into account, then they are not equal.

As the liquid column is in equilibrium, the forces acting on it are balanced. The vertical forces acting are

(i) Force P_1A acting vertically down on the top surface.
(ii) Weight mg of the liquid column acting vertically downwards.
(iii) Force P_2A at the bottom surface acting vertically upwards. where P_1 and P_2 are the pressures at the top and bottom faces, A is the area of cross section of the circular face and m is the mass of the
cylindrical liquid column.

$$\text{At equilibrium, } P_1A + mg - P_2A = 0 \text{ or } P_1A + mg = P_2A$$

$$P_2 = P_1 + mg\ A$$

$$\text{But } m = Ah\rho$$

$$\therefore P_2 = P_1 + Ah\rho gA$$

$$(i.e)\ P_2 = P_1 + h\rho g$$

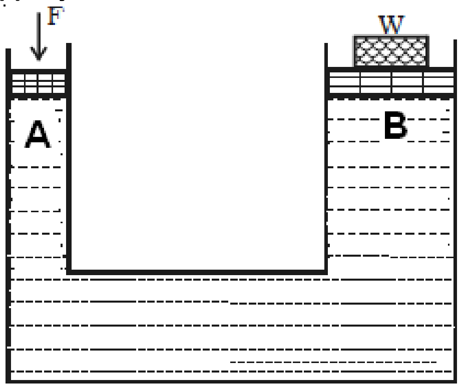
This equation proves that the pressure is the same at all points at the same depth. This results in another statement of *Pascal's law* which can be stated as *change in pressure at any point in an enclosed fluid at rest is transmitted undiminished to all points in the fluid and act in all directions.*

Characteristics of the fluid pressure

(i) Pressure at a point acts equally in all directions
(ii) Liquids at rest exerts lateral pressure, which increases with depth
(iii) Pressure acts normally on any area in whatever orientation the area may be held
(iv) Free surface of a liquid at rest remains horizontal
(v) pressure at every point in the same horizontal line is the same inside a liquid at rest
(vi) liquid at rest stands at the same height in communicating vessels

Application of Pascal's law

(i) Hydraulic lift



An important application of Pascal's law is the hydraulic lift used to lift heavy objects. A schematic diagram of a hydraulic lift is shown in the Fig.. It consists of a liquid container which has pistons fitted into the small and large opening cylinders. If a_1 and a_2 are the areas of the pistons A and B respectively, F is the force applied on A and W is the load on B, then

$$\frac{F}{a_1} = \frac{W}{a_2}$$

$$F = W\frac{a_1}{a_2}$$

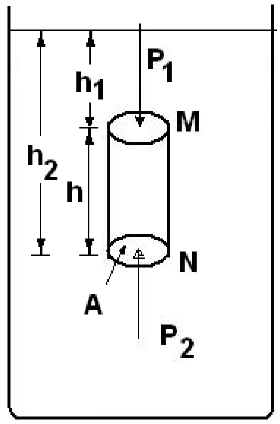
This is the load that can be lifted by applying a force F on A. In the above equation a_2/a_1 is called mechanical advantage of the hydraulic lift. One can see such a lift in many automobile service stations.

Buoyancy and Archimedes principle

If an object is immersed in or floating on the surface of a liquid, it experiences a net vertically upward force due to liquid pressure. This force is called as Buoyant force or force of Buoyancy and it acts from the centre of gravity of the displaced liquid. According to Archimedes principle, "the magnitude of force of buoyancy is equal to the weight of the displaced liquid"

To prove Archimedes principle, consider a body totally immersed in a liquid as shown in the figure.

The vertical force on the body due to liquid pressure may be found most easily by considering a cylindrical volume similar to that one shown in figure



The net vertical force on the element is

dF = (P_2- P_1) A

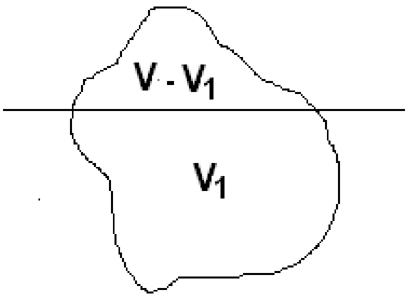
$$F = [(P_0 + h_2\rho g) - (P_0 + h_1\rho g)]A$$
$$F = (h_2 - h_1)\rho g A$$
$$F = h\rho g A$$

But volme V = hA

Thus F = Vρg

∴ force of Buoyancy = Vρg = Weight of liquid displaced

Expression for immersed volume of a floating Body



Let a solid of volume V and density ρ floats in liquid of density ρ_0. Volume V_1 of the body is immersed inside the liquid

The weight of floating body = Vρg

The weight of the displaced liquid = $V_1\rho_0 g$

For the body to float

Weight of body = Weight of liquid displaced

$V\rho g = V_1\rho_0 g$

$$\frac{V_1}{V} = \frac{\rho}{\rho_0}$$
$$V_1 = \frac{\rho V}{\rho_0}$$

∴ Immersed volume = mass of solid / density of liquid

From above it is clear that density of the solid volume must be less than density of the liquid to enable it to float freely in the liquid. How ever a metal vessel floats in water though the density of metal is much higher that the that of eater because floating bodies are hollow inside and hence displaces large volume. When thy float on water, the weight of the displaced water is equal to the weight of the body

Laws of floatation

The principle of Archimedes may be applied to floating bodies to give the laws of flotation

(i) When a body floats freely in a liquid the weight of the floating body is equal to the weight of the liquid displaced

(ii) The centre of gravity of the displaced liquid B (called the centre of buoyancy) lies vertically above or below the centre of gravity of the floating body G

Solved numerical

Q) A stone of mass 0.3kg and relative density 2.5 is immersed in a liquid of relative density 1.2. Calculate the resultant up thrust exerted on the stone by the liquid and the weight of stone in liquid

Solution

Volume of stone V = mass/density

V = 0.3/2.5 = 0.12 m^3

Upward thrust =buoyant force = $V\rho_0 g$ =0.12×1.2×9.8 = 1.41 N

Weight of stone in liquid = Gravitational force – buoyant force

=0.3×9.8 – 1.41 = 1.53 N or 0.156 kg wt

Q) A metal cube floats on mercury with (1/8) th of its volume under mercury. What portion of the cube will remain under mercury if sufficient water is added hust to cover the cube. Assume that the top surface of the cube remains horizontal in both cases. Relative density of mercury = 13.6

From the formula

$$V_1 = \frac{\rho V}{\rho_0}$$

Here V_1 is volume immersed in mercury = V/8 given and ρ_0 density of mercury, ρ density of metal

$$\frac{V}{8} = \frac{\rho V}{13.5}$$

ρ=1.725 is density of metal

Now let V' be the volume immersed in mercury then V-V' is volume immersed in water then

Weight of metal = Buoyant force due to Water + Buoyant force due to mercury

V(1.725) g= (V-V')×1×g +V'×13.6g

V(1.725) = (V-V')×1 + (V'×13.6)

V(0.725) = 12.6V'

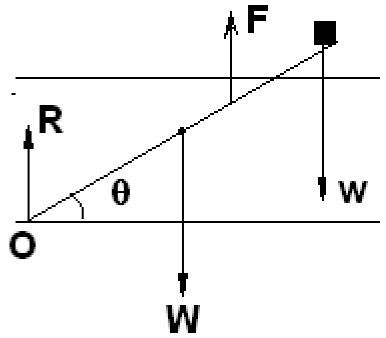
$$\frac{V'}{V} = \frac{0.725}{12.6} = \frac{1}{8}$$

Thus in the second case only (1/18)th of the volume of the cube is under mercury

Q) A rod of length 6 m has a mass of 12kg. If it is hinged at one end at a distance of 3m below a water surface

(i) What weight should be attached to the other end so that 5 m of rod be submerged?

(ii) find the magnitude and direction of the final force exerted on the rod exerted by hinge.
Specific gravity of the material of the rod is 0.5
Solution
Since one end is fixed in water we have to calculate moment of force



Moment of force due to weight of rod about point O= $Wg(L/2)\cos\theta$
Moment of force due to additional weight about point O =$wL\cos\theta$
Moment of force due to Buoyant force(F) about point O = $F(l/2)\cos\theta$
Here l is the length of rod immersed in water = 5m And L is total length of rod

Since rod is at rotational equilibrium at equilibrium
$F(l/2)\cos\theta = wL\cos\theta + Wg(L/2)\cos\theta$
$F(l/2) = wL + W(L/2)g$ --- eq(1)

But F = Vρg
Since 5m is immersed in water thus (5/6) of volume of rod is immersed
Volume of rod = mass/density = 12/0.5 = 24 m^3
Thus F = (5/6)×24×1×g = 20g N
Substituting values in eq(1) we get
∴ 20(2.5)g = w(6) + (12)(3)g
50 = 6w + 36
w = 14g/6 = 2.33g N
w = 2.33 kg wt
Now R = W+w-F
R = 12g +2.33g – 20g
R = -5.67g N
R = -5.67 kg wt
The negative sign indicates that the reaction (vertical) at the hinges acts downwards

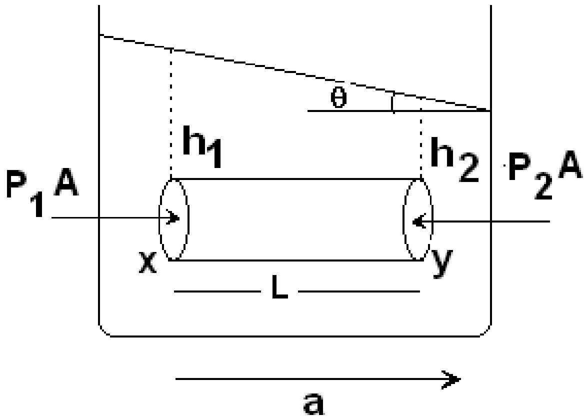
Liquid in accelerated Vessel

Variation of pressure and force of buoyancy in a liquid kept in accelerated vessel
Consider a liquid of density ρ kept in a vessel moving with acceleration **a in upward** direction. Let height of liquid column be h
Then effective gravitational acceleration on liquid = g +a
Thus pressured exerted at depth h $P = P_0 + \rho(g+a)h$
Similarly if liquid in container moves **down with acceleration a**

Then effective gravitational acceleration on liquid = g –a
Thus pressure exerted at depth h, $P = P_0 + \rho(g-a) h$

Also Buoyant force on immersed body when liquid is **moving up**
$F_B = V\rho(g+a)$
Buoyant force on immersed body when liquid is **moving down**
$F_B = V\rho(g+a)$
V is volume of the liquid displaced

Shape of free surface of a liquid in horizontal accelerated vessel



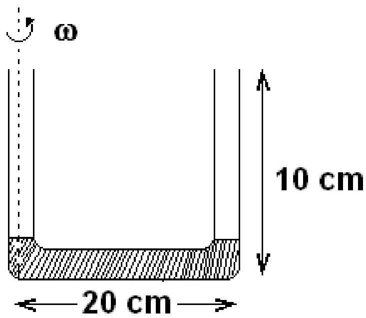
When a vessel filled with liquid accelerates horizontally. We observe its free surface inclined at some angle with horizontal. To find angle θ made by free surface with horizontal, consider a horizontal liquid column including two points x and y at the depth of h_1 and h_2 from the inclined free surface of liquid as shown in figure

Force on area at x = $P_1A = h_1\rho g$
Pseudo force at y= mass of liquid tube of length L and cross sectional area A ×acceleration
Pseudo force at y=$\rho(LA)$
Total force at y = P_1A + Pseudo force
Force on area at y = $h_1\rho g + \rho(LA)$
Since liquid is in equilibrium
Force on area at x = Force on area at y
$h_1\rho g = h_1\rho g + \rho(LA)$
$(h_1 - h_2) g = La$
From geometry of figure

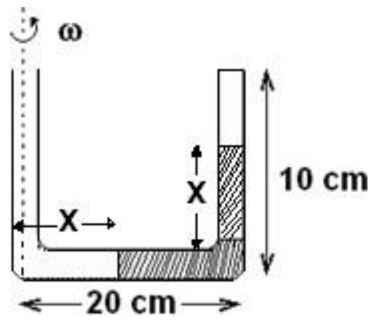
$$\frac{h_1 - h_2}{L} = \frac{a}{g}$$

$$tan\theta = \frac{a}{g}$$

Q) Length of a horizontal arm of a U-tube is 20cm and end of both the vertical arms are open to a pressure 1.01×10^3 N/m^2. Water is poured into the tube such that liquid just fills horizontal part of the tube is then rotated about a vertical axis passing through the other vertical arm with angular velocity ω. If length of water in sealed tube rises to 5cm, calculate ω. Take density of water = 10^3 kg/m^3 and g = 10 m/s^2. Assume temperature to be constant.



Solution



when tube is rotated liquid will experience a centrifugal force thus water moves up in second arm of the U tube.
When centrifugal force + pressure in first arm = force due to pressure in second closed arm +force due to liquid column then equilibrium condition is established ---eq(1)

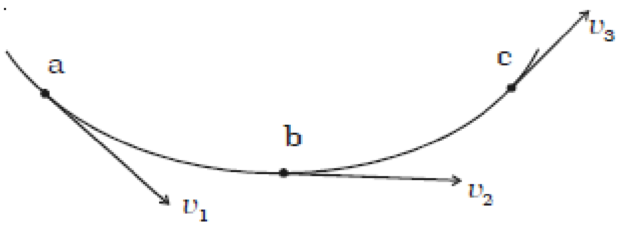
Calculation of force due to pressure in closed tube
Before closing pressure $P_i = 1.01\times10^3$ N/m^2
Volume before closing V_i= 0.1A (A is area of cross-section)
After closing the other arm Pressure P_f and volume V_f = 0.05A
From equation $P_iV_i = P_fV_f$
$(1.01\times10^3)\times 0.1A = P_f \times(0.05A)$
$P_f = 2.02\times10^3$
Force due to pressure = $(2.02\times10^3)\times A$
Pressure in first arm = 1.01×10^3
Calculation of force due to liquid column in second arm
Height of liquid column = 0.05 m
Thus pressure due to column = $h\rho g = 0.05\times10^3\times10$ =500 N/m^2
Force due to liquid column PA= 0.5A
Calculation of centrifugal force
Mass of the liquid in horizontal part = volume ×density = $(0.2-0.05)A\times10^3$=150A
Centre of mass of horizontal liquid from first arm 'r' $=0.05+\frac{0.2-0.05}{2}=0.125\ m$
Centrifugal force = $m\omega^2 r = 150A\times\omega^2\times0.125=(18.75A)\omega^2$

Now substituting values in equation 1 we get
$(18.75A)\times\omega^2+(1.01\times10^3)\times A=(2.02\times10^3)\times A + 500A$
$(18.75)\times\omega^2+1.01\times10^3=(2.02\times10^3) + 500$
ω = 8.97 rad/s

Fluid dynamics

Streamline flow

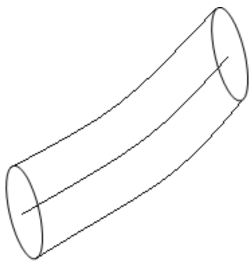
The flow of a liquid is said to be steady, streamline or laminar if every particle of the liquid follows exactly the path of its preceding particle and has the same velocity of its preceding particle at every point.



Let abc be the path of flow of a liquid and v_1, v_2 and v_3 be the velocities of the liquid at the points a, b and c respectively. During a streamline flow, all the particles arriving at 'a' will

have the same velocity v_1 which is directed along the tangent at the point 'a'. A particle arriving at b will always have the same velocity v_2. This velocity v_2 may or may not be equal to v_1.
Similarly all the particles arriving at the point 'c' will always have the same velocity v_3. In other words, in the streamline flow of a liquid, the velocity of every particle crossing a particular point is the same.
The streamline flow is possible only as long as the velocity of the fluid does not exceed a certain value. This limiting value of velocity is called critical velocity.

Tube of flow



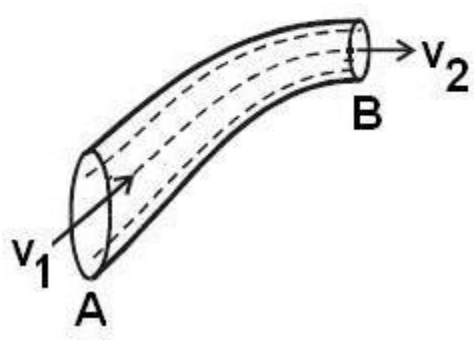
In a fluid having a steady flow, if we select a finite number of streamlines to form a bundle
like the streamline pattern shown in the figure, the tubular region is called a tube of flow.
The tube of flow is bounded by a streamlines so that by fluid can flow across the boundaries of the tube of flow and any fluid that enters at one end must leave at the other end.

Turbulent flow

When the velocity of a liquid exceeds the critical velocity, the path and velocities of the liquid become disorderly. At this stage, the flow loses all its orderliness and is called turbulent flow. Some examples of turbulent flow are :
(i) After rising a short distance, the smooth column of smoke
from an incense stick breaks up into irregular and random patterns.
(ii) The flash - flood after a heavy rain.
Critical velocity of a liquid can be defined as that velocity of liquid upto which the flow is streamlined and above which its flow becomes turbulent.

Equation of continuity

Consider a non-viscous liquid in streamline flow through a tube AB of varying cross section as shown in Fig. Let a_1 and a_2 be the area of cross section, v_1 and v_2 be the velocity of flow of the liquid at A and B respectively.

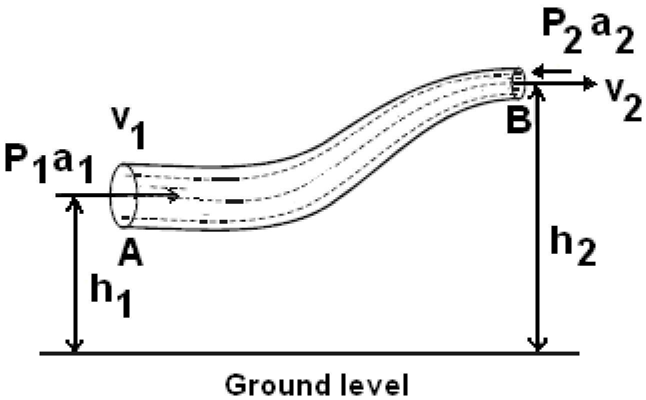


∴ Volume of liquid entering per second at A = a_1v_1.
If ρ is the density of the liquid, then mass of liquid entering per second at A = $a_1v_1\rho$.
Similarly, mass of liquid leaving per second at B = $a_2v_2\rho$
If there is no loss of liquid in the tube and the flow is steady, then mass of liquid entering per second at A = mass of liquid leaving per second at B
(i.e) $a_1v_1\rho = a_2v_2\rho$ or $a_1v_1 = a_2v_2$

i.e. av = constant
This is called as the equation of continuity. From this equation v is inversely proportional to area of cross-section along a tube of flow
i.e. the larger the area of cross section the smaller will be the velocity of flow of liquid and vice-versa.

Bernoulli's Equation

The theorem states that the work done by all forces acting on a system is equal to the change in kinetic energy of the system



Consider streamline flow of a liquid of density ρ through a pipe AB of varying cross section.
Let P_1 and P_2 be the pressures and a_1 and a_2, the cross sectional areas at A and B respectively. The liquid enters A normally with a velocity v_1 and leaves B normally with a velocity v_2. The liquid is accelerated against the force of gravity while flowing from A to B, because the height of B is greater than that of A from the ground level. Therefore P_1 is greater than P_2. This is maintained by an external force.
The mass m of the liquid crossing per second through any section of the tube in accordance with the equation of continuity is $a_1v_1\rho = a_2v_2\rho = m$
Or

$$a_1v_1 = a_2v_2 = \frac{m}{\rho}$$

As $a_1 > a_2$, $v_1 < v_2$
The force acting on the liquid at A = P_1a_1
The force acting on the liquid at B = $P_2 a_2$
Work done per second on the liquid at A = $P_1a_1 \times v_1 = P_1V$
Work done by the liquid at B = $P_2a_2 \times v_2 = P_2V$
∴ Net work done per second on the liquid by the pressure energy in moving the liquid from A to B is = $P_1V - P_2V$
If the mass of the liquid flowing in one second from A to B is m, then increase in potential energy per second of liquid from A to B is = $mgh_2 - mgh_1$
Increase in kinetic energy per second of the liquid.

$$\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

According to work-energy principle, work done per second by the pressure energy = (Increase in potential energy + Increase in kinetic energy) per second

$$P_1V - P_2V = (mgh_2 - mgh_1) + (\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2)$$

$$P_1V + mgh_1 + \frac{1}{2}mv_1^2 = P_2V + mgh_2 + \frac{1}{2}mv_2^2$$

$$\frac{P_1V}{V} + \frac{m}{V}gh_1 + \frac{1}{2}\frac{m}{V}v_1^2 = \frac{P_2V}{V} + \frac{m}{V}gh_2 + \frac{1}{2}\frac{m}{V}v_2^2$$

$$P_1 + \rho gh_1 + \frac{1}{2}\rho v_1^2 = P_2 + \rho gh_2 + \frac{1}{2}\rho v_2^2$$

Since subscripts 1 and 2 refer to any location on the pipeline, we can write in general

$$P + \rho gh + \frac{1}{2}mv^2 = constant$$

The above equation is called Bernoulli's equation for steady non-viscous incompressible flow. Dividing the above equation by gh we can rewrite the above equation as

$$h + \frac{v^2}{2g} + \frac{P}{\rho g} = constnat, \text{ which is called total head}$$

Term h is called elevation head or gravitational head

$$\frac{v^2}{2g} \text{ is called velocity head}$$

$$\frac{P}{\rho g} \text{ is called pressure head}$$

Above equation indicates for **ideal liquid velocity increases when pressure decreases and vice-versa**

Q) A vertical tube of diameter 4mm at the bottom has a water passing through it. If the pressure be atmospheric at the bottom where the water emerges at the rate of 800gm per minute, what is the pressure at a point in the tube 5cm above the bottom where the diameter is 3mm

Solution

Rate of flow of water = 800 gm/min = (40/3)gm/sec

Now mass of water per sec = velocity ×area × density

$40/3 = V_1 \times [\pi (0.2)^2] \times 1$

$V_1 = (333.33/\pi)$ cm/sec

Now $A_1V_1 = A_2V_2$ Thus

$V_2 = (4/3)V_1$

$V_2 = (444.44/\pi)$ cm/sec is the velocity at height 25cm

Now $P_1 =$ atmospheric pressure $= 1.01\times10^7$ dyne

Now from Bernoulli's equation

$$P_1 + \rho gh_1 + \frac{1}{2}\rho v_1^2 = P_2 + \rho gh_2 + \frac{1}{2}\rho v_2^2$$

$$1.01 \times 10^6 + 0 + \frac{1}{2}\left(\frac{333.33}{\pi}\right)^2 = P_2 + 1 \times 981 \times 25 + \frac{1}{2}\left(\frac{444.44}{\pi}\right)^2$$

On solving

$P_2 = 0.98\times10^6$ dyne

Now pressure = $h\rho_0 g$ here ρ_0 is density of mercury = 13.6 in cgs system

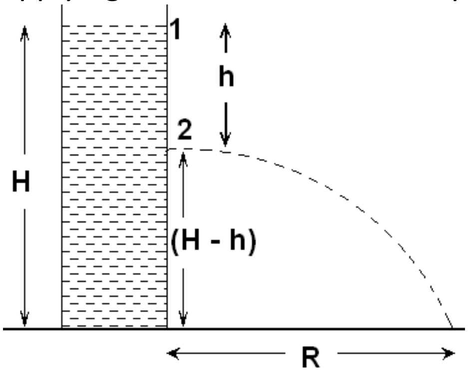
$0.98\times10^6 = h\times980\times13.6$

H = 73.5 cm of Hg

Q) Water stands at a depth H in a tank whose side walls are vertical. A hole is made at one of the walls at depth h below the water surface. Find at what distance from the foot of the wall does the emerging stream of water strike the flower. What is the maximum possible range?

Solution

Applying Bernoulli's theorem at point 1 and 2



$$P_1 + \rho g h_1 + \frac{1}{2}\rho v_1^2 = P_2 + \rho g h_2 + \frac{1}{2}\rho v_2^2$$

$P_1 = P_2 = P$ (atmospheric pressure)

$V_1 = 0$, $h_2 = H-h$ and $h_1 = H$

$$\rho g H = \rho g (H - h) + \frac{1}{2}\rho v_2^2$$

$$v_2^2 = 2gh$$

The vertical component of velocity of water emerging from hole at 2 is zero. Therefore time taken (t) by the water to fall through a distance (H-h) is given bu

$$H - h = \frac{1}{2} g t^2$$

$$t = \sqrt{\frac{2(H-h)}{g}}$$

Required horizontal range $R = v_2 t$

$$R = \sqrt{2gh}\sqrt{\frac{2(H-h)}{g}}$$

$$R = 2\sqrt{h(H-h)}$$

the range is maximu when dR/dh = 0

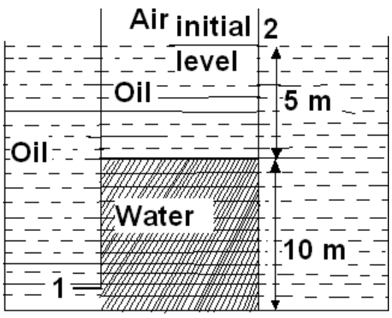
$$2 \times \frac{1}{2}(Hh - h^2)^{\frac{-1}{2}}(H - 2h) = 0$$

This gives h = H/2

Therefore Maximu range =

$$R = 2\sqrt{\frac{H}{2}\left(H - \frac{H}{2}\right)} = H$$

Q) A tank with a small circular hole contains oil on top of water. It is immersed in a large tank of same oil. Water flows through the hole. What is the velocity of the flow initially? When the flow stops, what would be the position of the oil-water interface in the tank? The ratio of the cross-section area tank to the that of hole is 50, determine the time at which the flow stops, density of oil = 800 kg/m³



Solution:

Pressure at hole and pressure at point on the bottom of water is different thus water flows through the hole

Pressure at point 1 $P_1 = P_0 + h\rho_0 g$ here h = 15m and $\rho_0 = 800\ kg/m^3$

Pressure at point 2 is $P_2 = P_0$

And potential = $5\rho_0 g + 10\rho g$ here ρ is density of water

$$P_1 + \rho g h_1 + \frac{1}{2}\rho v_1^2 = P_2 + \rho g h_2 + \frac{1}{2}\rho v_2^2$$

$$P_0 + h\rho_0 g + \frac{1}{2}\rho v_1^2 = P_0 + 5\rho_0 g + 10\rho g + \frac{1}{2}\rho v_2^2$$

For continuity equation $A_1 V_1 = A_2 V_2$

$$V_2 = \frac{A_1}{A_2} V_1 = \frac{1}{50} V_1$$

$$15 \times 800 \times 10 + \frac{1}{2} 1000 v_1^2 = 5 \times 800 \times 10 + 10 \times 1000 \times 10 + \frac{1}{2} 1000 \times \left(\frac{v_1}{50}\right)^2$$

$$120000 + 500 v_1^2 = 140000 + 500 \times \left(\frac{v_1}{50}\right)^2$$

$$v_1^2\left(500 - \frac{1}{2500}\right) = 20000$$

$$v_1^2(500) = 20000$$

$V_1 = 6.32$ m/s

Let x be the height of water column when flow of water is stopped

Applying Bernoull's equation between point a and x we het

$$P_0 + 15\rho_0 g + \frac{1}{2}\rho v_1^2 = P_0 + 5\rho_0 g + x\rho g + \frac{1}{2}\rho v_2^2$$

Since velocities are zero

$$15\rho_0 g = 5\rho_0 g + x\rho g$$

$$15 \times 800 = 5 \times 800 + x \times 1000$$

X = 8 m

Let at any moment of time height of water column be y then level of oil in samll tank is (15-y) accoding to bernolli's equation

$$P_0 + 15\rho_0 g + \frac{1}{2}\rho v_1^2 = P_0 + (5)\rho_0 g + y\rho g + \frac{1}{2}\rho v_2^2$$

$$\frac{1}{2}\rho v_1^2 = (-10)\rho_0 g + y\rho g + \frac{1}{2}\rho v_2^2$$

$av_1 = Av_2$

$$v_2 = \frac{a}{A} v_1 = \frac{v_1}{50}$$

$$\frac{1}{2}\rho v_1^2 = (-10)\rho_0 g + y\rho g + \frac{1}{2}\rho\left(\frac{v_1}{50}\right)^2$$

Neglecting term

$$\frac{1}{2}\rho\left(\frac{v_1}{50}\right)^2$$

$$\frac{1}{2}\rho v_1^2 = (-10)\rho_0 g + y\rho g$$

$$\frac{1}{2}1000 v_1^2 = (-10) \times 800 \times 10 + y \times 1000 \times 10$$

$$v_1^2 = -160 + 20y$$

Differentiating

$$2v_1 \frac{dv_1}{dt} = 20\frac{dy}{dt}$$

But

$$\frac{dy}{dt} = v_2 = -\frac{v_1}{50}$$

Negative sign since vleocity is decreasing

$$v_1 \frac{dv_1}{dt} = 10 \times \frac{-v_1}{50}$$

$$\frac{dv_1}{dt} = \frac{-1}{5}$$

$$dv_1 = \frac{-1}{5} dt$$

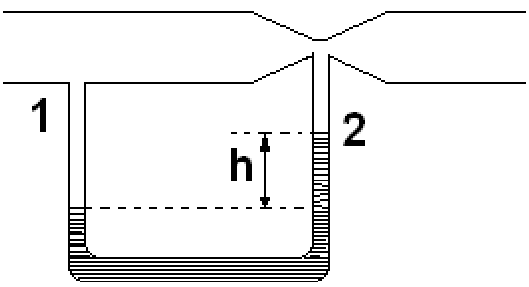
Integrating

$$\int_{6.32}^{0} dv_1 = \frac{-1}{5}\int_0^t dt$$

$$-6.23 = \frac{-t}{5}$$

t = 31.15 sec

Venturimeter:



This is a device based on Bernoulli's principle used for measuring the flow of a liquid in pipes. A liquid of density ρ flows through a pipe of cross-sectional area A. Let the constricted part of the cross-sectional area be 'a'. A manometer tube with a liquid say mercury having a densityρ_0 is attached to the tube as shown in figure

If P_1 is the pressure at point 1 and P_2 the pressure at point 2, we have

$$P_1 + \frac{1}{2} m v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

Where v_1 and v_2 are the velocities at these points respectively

$$\frac{1}{2} v_1^2 - \frac{1}{2} v_2^2 = \frac{P_2}{\rho} - \frac{P_1}{\rho}$$

We have $Av_1 = av_2$

$$v_2 = \frac{A}{a} v_1$$

$$\frac{1}{2} v_1^2 - \frac{1}{2}\left(\frac{A}{a}\right)^2 v_1^2 = \frac{P_2 - P_1}{\rho}$$

$$v_1^2 - \left(\frac{A}{a}\right)^2 v_1^2 = \frac{2(P_2 - P_1)}{\rho}$$

$$v_1^2\left(1 - \frac{A^2}{a^2}\right) = \frac{2(P_2 - P_1)}{\rho}$$

$$v_1^2 = \frac{\frac{2(P_2 - P_1)}{\rho}}{1 - \frac{A^2}{a^2}} = \frac{2a^2(P_2 - P_1)}{(a^2 - A^2)\rho}$$

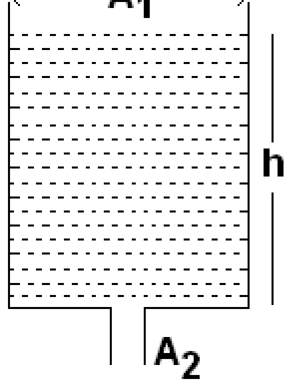
$$v_1 = \sqrt{\frac{2a^2(P_2 - P_1)}{(a^2 - A^2)\rho}}$$

Volume of liquid flowing through the pipe per second $Q = Av_1$

$$Q = Aa\sqrt{\frac{2(P_2 - P_1)}{(a^2 - A^2)\rho}}$$

Speed of Efflux

As shown in figure a tank of cross-sectional area A, filled to a depth h with a liquid of density ρ. There is a hole of cross-section area A_2 at the bottom and the liquid flows out of the tank through the hole $A_2 << A_1$



Let v_1 and v_2 be the speeds of the liquid at A_1 and A_2. As both the cross sections are opened to the atmosphere, the pressure there equals to atmospheric pressure P_o. If the height of the free surface above the hole is h_1

Bernoulli's equation gives

$$P_0 + \frac{1}{2}\rho v_1^2 + \rho g h = P_0 + \frac{1}{2}\rho v_2^2$$

By the equation of continuity,

$A_1 v_1 = A_2 v_2$

$$P_0 + \frac{1}{2}\rho\left(\frac{A_2}{A_1}\right)^2 v_2^2 + \rho g h = P_0 + \frac{1}{2}\rho v_2^2$$

$$\left[1 - \left(\frac{A_2}{A_1}\right)^2\right] v_2^2 = 2gh$$

$$v_2 = \sqrt{\frac{2gh}{1 - \left(\frac{A_2}{A_1}\right)^2}}$$

If $A_2 <<< A_1$, the equation reduces to $v_2 = \sqrt{(2gh)}$

The speed of efflux is the same as the speed a body that would acquire in falling freely through a height h. This is known as Torricelli's theorem.

SURFACE TENSION & VISCOSITY

SURFACE TENSION

Intermolecular forces

The force between two molecules of a substance is called intermolecular force. This intermolecular force is basically electric in nature. When the distance between two molecules is greater, the distribution of charges is such that the mean distance between opposite charges in the molecule is slightly less than the distance between their like charges. So a force of attraction exists. When the intermolecular distance is less, there is overlapping of the electron clouds of the molecules resulting in a strong repulsive force. The intermolecular forces are of two types. They are (i) cohesive force and (ii) adhesive force.

Cohesive force

Cohesive force is the force of attraction between the molecules of the same substance. This cohesive force is very strong in solids, weak in liquids and extremely weak in gases.

Adhesive force

Adhesive force is the force of attraction between the molecules of two different substances. For example due to the adhesive force, ink sticks to paper while writing. Fevicol, gum etc exhibit strong adhesive property.

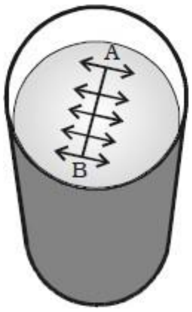
Water wets glass because the cohesive force between water molecules is less than the adhesive force between water and glass molecules. Whereas, mercury does not wet glass because the cohesive force between mercury molecules is greater than the adhesive force between mercury and glass molecules.

Molecular range and sphere of influence

Molecular range is the maximum distance upto which a molecule can exert force of attraction on another molecule. It is of the order of 10^{-9} m for solids and liquids.

Sphere of influence is a sphere drawn around a particular molecule as centre and molecular range as radius. The central molecule exerts a force of attraction on all the molecules lying within the sphere of influence.

Surface tension of a liquid



Surface tension is the property of the free surface of a liquid at rest to behave like a stretched membrane in order to acquire minimum surface area.

Imagine a line AB in the free surface of a liquid at rest (Fig. 5.20). The force of surface tension is measured as the force acting per unit length on either side of this imaginary line AB. The force is perpendicular to the line and tangential to the liquid surface. If F is the force acting on the length *l* of the line AB, then surface tension is given by

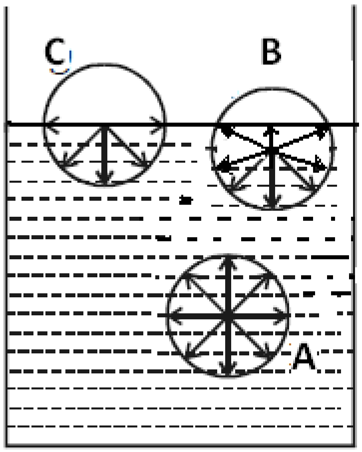
$$T = \frac{F}{L}.$$

Surface tension is defined as the force per unit length acting perpendicular on an imaginary line drawn on the liquid surface, tending to pull the surface apart along the line. Its unit is N m^{-1} and dimensional formula is MT^{-2}.

It depends on temperature. The surface tension of all liquids decreases linearly with temperature

It is a scalar quantity and become zero at critical temperature

Molecular theory of surface tension

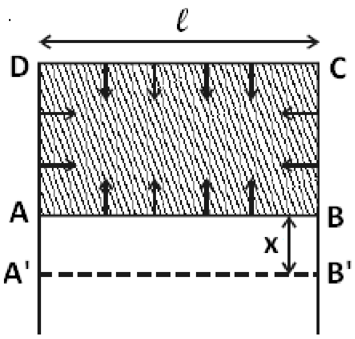


The surface tension of liquid arises out of the attraction of its molecules. Molecules of fluid (liquid and gas) attract one another with a force. If any other molecule is within the sphere of influence of first molecule it will experience a force of attraction

Consider three molecules A, B, C having their spheres of influence as shown in the figure. The sphere of influence of A is well inside the liquid, that of B partly outside and that of C exactly half of total

Molecules like A do not experience any resultant force, as they are attracted equally in all directions. Molecules like B or C will experience a resultant force directed inward. Thus the molecules will inside the liquid will have only kinetic energy but the molecule near surface will have kinetic as well as potential energy which is equal o the work done in placing them near the surface against the force of attraction directed inward

Surface energy



Any Strained body possesses potential energy, which is equal to the work done in bringing it to the present state from its initial unstained state. The surface of liquid is also a strained system and hence the surface of a liquid also has potential energy, which is equal to the work done increasing the surface. This energy per unit area of the surface is called surface energy

To derive an expression for surface energy consider a wire frame equipped with a sliding wire AB as shown in figure. A film of soap solution is formed across ABCD of the frame. The side AB is pulled to the left due to surface tension. To keep the wire in position a force F has to be applied to the right. If T is the surface tension and l is the length of AB, then the force due to surface tension over AB is 2lT to the left because the film has two surfaces (upper and lower)

Since the film is in equilibrium F = 2lT

Now, if the wire AB is pulled down, energy will flow from the agent to the film and this energy is stored as potential energy of the surface created just now. Let the wire be pulled slowly through x.
Then the work done = energy added to the film from above agent
W=Fx = 2lTx
Potential energy per unit area (surface energy) of the film

$$U = \frac{2lTx}{2lx} = T$$

$$T = \frac{W}{area}$$

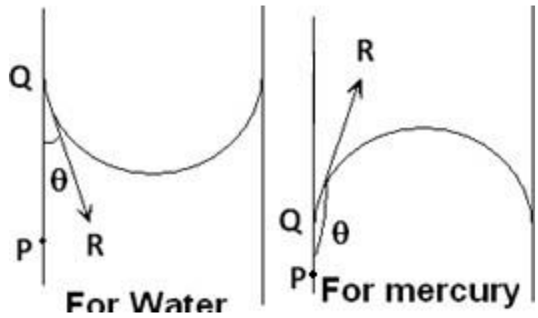
Thus surface energy numerically equal to its surface tension
It s unit is Joule per square metre (Jm^{-2})

Solved Numerical

Q) Calculate the work done in blowing a soap bubble of radius 10cm, surface tension being 0.08 Nm^{-1}. What additional work will be done in further blowing it so that its radius is doubled?
Solution
In case of a soap bubble, there are two free surfaces
Surface tension = Work done per unit area
∴ Work done in blowing a soap bubble of radius R is given by = Surface tension × Area
W = T ×$(2\times4\pi R^2)$
W = (0.06)×$(8\times3.14\times0.1^2)$
W = 1.51 J
Similarly, work done in forming a bubble of radius 0.2 m is
W' = (0.06)×$(8\times3.14\times0.2^2)$ = 60.3 J
Additional work done in doubling the radius of the bubble is given by
W' – W = 60.3 – 1.51 = 5.42J

Q) A mercury drop of radius 1cm is sprayed into 10^6 droplets of equal size. Calculate the energy expended if surface tension of mercury is 35×10^{-3} N/m
Solution
Since total volume of 10^6 droplet has remains same
If radius small droplet is r' and big drop is r then r = $(10^6)^{1/3}$ r'
1 = 10^2r' or r' = 0.01 cm = 10^{-4} m
Since surface area is increased energy should be supplied to make small small drops
Total energy of small droplet =[T $(4\pi r'^2)$] 10^6
Total energy of big droplet = [T$(44\pi r^2)$]
Spending of energy = Total energy of small droplets - Total energy of big droplet
Spending of energy = [T $(4\pi r'^2)$] 10^6 - [T$(4\pi r^2)$]
Spending of energy = T×4π [$10^6\times r'^2 - r^2$]
Spending of energy = $35\times10^{-3}\times4\times3.14$ [$10^6\times(10^{-4})^2 - (10^{-2})^2$]
Spending of energy =0.44[$10^{-2} - 10^{-4}$]
Spending of energy = 4.356×10^{-3} J

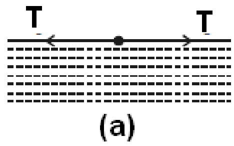
Angle of contact

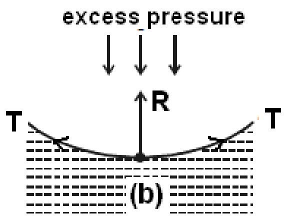


When the free surface of a liquid comes in contact with a solid, it becomes curved at the point of contact. *The angle between the tangent to the liquid surface at the point of contact of the liquid with the solid and the solid surface inside the liquid is called angle of contact.* In Fig., QR is the tangent drawn at the point of contact Q. The angle PQR is called the angle of contact. When a liquid has concave meniscus, the angle of contact is acute. When it has a convex meniscus, the angle of contact is obtuse. The angle of contact depends on the nature of liquid and solid in contact. For water and glass, θ lies between 8^o and 18^o. For pure water and clean glass, it is very small and hence it is taken as zero. The angle of contact of mercury with glass is 138^o.

Pressure difference across a liquid surface

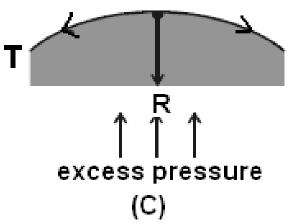
If the free surface of a liquid is plane, then the surface tension acts horizontally (Fig. a). It has no component perpendicular to the horizontal surface. As a result, there is no pressure difference between the liquid side and the vapour side.





If the surface of the liquid is concave (Fig. b), then the resultant force *R* due to surface tension on a molecule on the surface act vertically upwards. To balance this, an excess of pressure acting downward on the concave side is necessary.

On the other hand if the surface is convex (Fig.c), the resultant *R* acts downward and there must be an excess of pressure on the concave side acting in the upward direction.



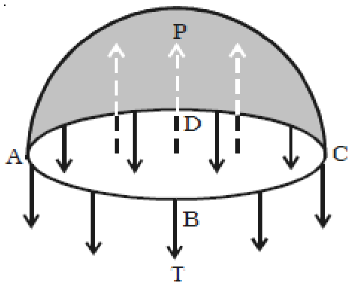
Thus, there is always an excess of pressure on the concave side of a curved liquid surface over the pressure on its convex side due to surface tension.

Excess pressure

The pressure inside a liquid drop or a soap bubble must be in excess of the pressure outside the bubble or drop because without such pressure difference a drop or a bubble cannot be in state of equilibrium. Due to surface tension the drop or bubble has got the tendency to contract and disappear altogether.

To balance this, there must be an excess of pressure inside the bubble.

To obtain a relation between the excess pressure and the surface tension, consider a water drop of radius r and surface tension T, The excess of pressure P inside the drop provides a force acting outwards perpendicular to the surface, to balance the resultant force due to surface tension.



Imagine the drop to be divided into two equal halves. Considering the equilibrium of the upper hemisphere of the drop, the upward force on the plane face ABCD due to excess pressure P is $P \pi r^2$

If T is the surface tension of the liquid, the force due to surface tension acting downward along the circumference of the circle ABCD is $T\,2\pi r$.

At equilibrium, $P\pi r^2 = T\,2\pi r$

$$P = \frac{2T}{r}$$

Here P is excess pressure $P = P_i - P_O$

$$P_i - P_O = \frac{2T}{r}$$

Excess pressure inside a soap bubble

A soap bubble has two liquid surfaces in contact with air, one inside the bubble and the other outside the bubble. Therefore the force due to surface tension = $2 \times 2\pi rT$

∴ At equilibrium, $P\pi r^2 = 2 \times 2\pi rT$

$$P = \frac{4T}{r}$$

Thus the excess of pressure inside a drop is inversely proportional to its radius

the pressure needed to form a very small bubble is high. This explains why one needs to blow hard to start a balloon growing. Once the balloon has grown, less air pressure is needed to make it expand more.

Solved Numerical

Q) An air bubble of radius R is formed on a narrow tube having a radius r where R>>r. Air of density ρ is blown inside the tube with velocity V. The air molecules collide perpendicularly with the wall of bubble and stop. Find the radius at which the bubble separates from the tube. Take surface tension of bulb as T

Solution:

Air molecules collides at stops thus force exerted on the soap bubble

Mass of air = Volume ×ρ

Volume of air = velocity of air × area of hole = $v(\pi r^2)$

Mass of air = $v\,\rho(\pi r^2)$

Force exerted by the air = change in momentum of air molecules

Force due to air molecule = $(v\,\rho\pi r^2)\,v = \rho\pi r^2 v^2$

Pressure of blown air in side the bubble = ρv^2

Now Force due to surface tension of bubble of radius R

Pressure difference in bubble = 4T/R

Bubble gets separated when pressure difference in bubble = pressure of blown air

$$\frac{4T}{R} = \rho v^2$$

$$R = \frac{4T}{\rho v^2}$$

Q) Two spherical soap bubbles coalesce to form a single bubble. If V is the consequent change in volume of the contained air and S the change in the total surface area, show that 3PV+4ST = 0, where T is the surface tension of the soap bubble and P the atmospheric pressure

Solution:

$$P_1 = P + \frac{4T}{r_1}\ ;\ P_2 = P + \frac{4T}{r_2}$$

Since the total number of moles remains same

$N_1 + n_2 = n$

$P_1V_1 + P_2V_2 = P_3V_3$

$$\left(P + \frac{4T}{r_1}\right)\left(\frac{4}{3}\pi r_1^3\right) + \left(P + \frac{4T}{r_2}\right)\left(\frac{4}{3}\pi r_2^3\right) = \left(P + \frac{4T}{r}\right)\left(\frac{4}{3}\pi r^3\right)$$

$$\left(P + \frac{4T}{r_1}\right)(r_1^3) + \left(P + \frac{4T}{r_2}\right)(r_2^3) = \left(P + \frac{4T}{r}\right)(r^3)$$

$$Pr_1^3 + 4Tr_1^2 + Pr_2^3 + 4Tr_2^2 = Pr^3 + 4Tr^2$$

$$4Tr_1^2 + 4Tr_2^2 - 4Tr^2 = Pr^3 - Pr_1^3 - Pr_2^3$$

$$4T(r_1^2 + r_2^2 - r^2) = P(r^3 - r_1^3 - r_2^3)$$

$$\frac{4}{3}\pi 4T(r_1^2 + r_2^2 - r^2) = \frac{4}{3}\pi P(r^3 - r_1^3 - r_2^3)$$

$$4T(S_1 + S_2 - S_3) = 3P(V_3 - V_1 - V_3)$$

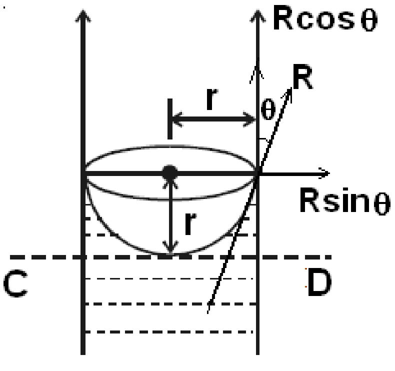
$$4TS = -3PV$$

Negative V because $V_3 < V_1 + V_2$

$$4TS + 3PV = 0$$

Surface tension by capillary rise method

Let us consider a capillary tube of uniform bore dipped vertically in a beaker containing water. Due to surface tension, water rises to a height *h* in the capillary tube as shown in Fig.. The surface tension *T* of the water acts inwards and the reaction of the tube *R* outwards. *R* is equal to *T* in magnitude but opposite in direction. This reaction *R* can be resolved into two rectangular components.



(i) Horizontal component *R sin* θ acting radially outwards

(ii) Vertical component *R cos* θ acting upwards.

The horizontal component acting all along the circumference of the tube cancel each other whereas the vertical component balances the weight of water column in the tube.

Total upward force = *R cos* θ × circumference of the tube

$F = 2\pi r\, R \cos\theta$ or $F = 2\pi r\, T \cos\theta$ ...(1)

[∵ $R = T$]

This upward force is responsible for the capillary rise. As the water column is in equilibrium, this force acting upwards is equal to weight of the water column acting downwards.

(i.e) $F = W$ ...(2)

Now, volume of water in the tube is assumed to be made up of

(i) a cylindrical water column of height h and (ii) water in the meniscus above the plane CD.

Volume of cylindrical water column = $\pi r2h$

Volume of water in the meniscus

= (Volume of cylinder of height r and radius r) – (Volume of hemisphere)

∴Volume of water in the meniscus=

$$\pi r^2 \times r - \frac{2}{3}\pi r^3 = \frac{1}{3}\pi r^3$$

∴Total volume of water in the tube

$$\pi r^2 h + \frac{1}{3}\pi r^3 = \pi\ ^2 (h + \frac{r}{3})$$

If ρ is the density of water, then weight of water in the tube is

$$W = \pi r^2 (h + \frac{r}{3}) \rho g \quad --- eq(3)$$

Substituting (1) and (3) in (2),

$$\pi r^2 (h + \frac{r}{3}) \rho g = 2\pi r T cos\theta$$

$$T = \frac{\pi r^2 (h + \frac{r}{3}) \rho g}{2\pi r cos\theta}$$

Since r is very small, r/3 can be neglected compared to h.

$$T = \frac{hr\rho g}{2cos\theta}$$

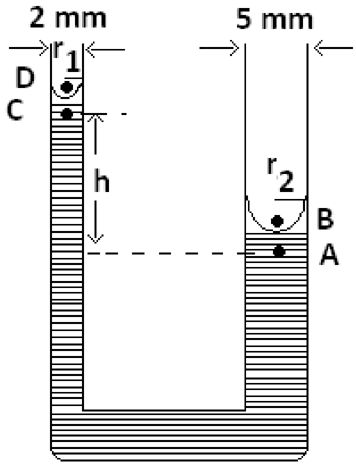
For water θ is very small cosθ =1

$$T = \frac{hr\rho g}{2}$$

Solved Numerical

Q) An U-tube with limbs of diameter 5mm and 2mm contains water of surface tension 7×10^{-2} N/m,angle of contact zero and density 1×10^{3} kg/m^3. Find the difference in levels (g= 10 m/s^2)

Solution: If the menisci are spherical, they will be hemispheres Since angle of contact is zero, their radii will then equal to radii of the limbs. The pressure on the concave side of



each surface exceeds that on the convex side by 2T/r, where T is surface tension and r is the radius of the limb concerned
Now $r_1 = 2.5$ mm $= 2.5\times10^{-3}$ m and $r_2 = 1$mm $= 10^{-3}$m
Hence

$$P_B - P_A = \frac{2T}{r_2} = \frac{2\times7\times10^{-2}}{2.5\times10^{-3}} = 56\ Pa$$

$P_A = P_B + 56 = P + 56$
Similarly

$$P_D - P_C = \frac{2T}{r_1} = \frac{2\times7\times10^{-2}}{10^{-3}} = 140\ Pa$$

$P_D = P_C + 140 = P + 140$

Since $P_D = P_B = P$

$\therefore P_A - P_C = (P-56) - (P-140)$

$P_A - P_C = 84$ Pa

But $P_A = P_C + h\rho g$

$H\rho g = 84$ Pa

$$\therefore h = \frac{84}{10^3\times10} = 8.4\ mm$$

Q) A mercury barometer has a glass tube with an inside diameter equal to 4mm. Since the contact angle of mercury with glass is 140°, capiliary depresses the column. How many millimeters of mercury must be added to the reading to correct for capillarity (Assume surface tension of mercury T = 0.545 N/m, density of mercury = 13.6×10^3)
Solution:
The height difference due to capillarity give by

$$h = \frac{2T\cos\theta}{r\rho g}$$

$$h = \frac{2\times0.545\times\cos140}{(2\times10^{-3})(13.6\times10^3)(9.8)} = -0.0031m$$

Therefore 3.1mm must be added to the barometer reading

Factors affecting surface tension
Impurities present in a liquid appreciably affect surface tension. A highly soluble substance like salt increases the surface tension whereas sparingly soluble substances like soap decreases the surface tension.
The surface tension decreases with rise in temperature. The temperature at which the surface tension of a liquid becomes zero is called critical temperature of the liquid.
Applications of surface tension
(i) During stormy weather, oil is poured into the sea around the ship. As the surface tension of oil is less than that of water, it spreads on water surface. Due to the decrease in surface tension, the velocity of the waves decreases. This reduces the wrath of the waves on the ship.

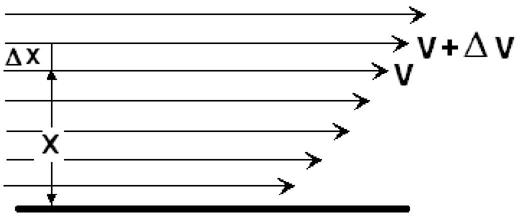
(ii) Lubricating oils spread easily to all parts because of their low surface tension.
(iii) Dirty clothes cannot be washed with water unless some detergent is added to water. When detergent is added to water, one end of the hairpin shaped molecules of the detergent get attracted to water and the other end, to molecules of the dirt. Thus the dirt is suspended surrounded by detergent molecules and this can be easily removed. This detergent action is due to the reduction of surface tension of water when soap or detergent is added to water.
(iv) Cotton dresses are preferred in summer because cotton dresses have fine pores which act as capillaries for the sweat.

VISCOSITY

If we pour equal amounts of water and castor oil in two identical funnels. It is observed that water flows out of the funnel very quickly whereas the flow of castor oil is very slow. This is because of the frictional force acting within the liquid. This force offered by the adjacent liquid layers is known as viscous force and the phenomenon is called viscosity. *Viscosity is the property of the fluid by virtue of which it opposes relative motion between its different layers. Both liquids and gases exhibit viscosity but liquids are much more viscous than gases.*

Co-efficient of viscosity

Consider the slow and steady flow of a fluid over a fixed horizontal surface as shown in the Fig. Let v be the velocity of thin layer of liquid at a distance x from the fixed solid surface.



Then according to Newton, the viscous force acting tangentially to the layer is proportional to the area of the layer and the velocity gradient at the layer. If F is the viscous force on the layer then ,
$F \propto A$, where A is the area of the layer and

$$F \propto -\frac{\Delta v}{\Delta t}$$

The negative sign is put to account for the fact that the viscous force is opposite to the direction of motion Thus

$$F = -\eta A \frac{dv}{dt}$$

Where η is a constant depending upon the nature of the liquid and is called the coefficient of viscosity and

$$velocity\ gradiant = \frac{dv}{dt}$$

If A = 1 and dv/dx = 1. We have F=- η

Thus the coefficient of viscosity of a liquid may be defined as the viscous force per unit area of the layer where velocity gradient is unity

The coefficient of viscosity has the dimension $[ML^{-1}T^{-1}]$ and its unit is Newton second per square metre (Nsm^{-2}) or kilogram per metre per second ($kgm^{-1}s^{-}$). In CGS, the unit of viscosity is Poise, 1kilogram per metre per second = 10 Poise

Stroke’s Law

When a solid moves through a viscous medium, its motion is oppsed by a viscous force depending on the velocity and shape and size of the body. The energy of the body is continuously decreases in overcoming the viscous resistance of the medium. This is why cars, aeroplanes etc are shaped streamline to minimize the viscous resistance on them
The viscous drag on a spherical body of radius r, moving with velocity v, in a viscous medium of viscosity η is given by

$$F_{viscous} = 6\pi\eta rv$$

This relation is called **Stoke’s law**
This law can be deduced by the method of dimensions.

Terminal Velocity

Let the body be driven by a constant force. In the beginning velocity v = 0 and acceleration ‘a’ is max so the body experiences small viscous force. With increase in speed viscous force goes on increasing till resultant force acting on the body becomes zero, and body moves with constant speed , this speed is known as terminal velocity
Consider the downward movement of a spherical body through a viscous medium such as a ball falling through a viscous medium as a ball falling through a liquid. If r is the radius of the body, ρ the density of the material of the body and σ is the density of the liquid, then
(i)The weight of the body down ward force

$$\frac{4}{3}\pi r^3 \rho g$$

(ii) The buoyancy of the body upward force

$$\frac{4}{3}\pi r^3 \rho_0 g$$

Net down ward force

$$\frac{4}{3}\pi r^3 (\rho - \rho_0) g$$

If v is the terminal velocity of the body , then viscous force $F_{viscous} = 6\pi\eta rv$

When acceleration becomes zero
upward viscous force = resultant down ward force

$$6\pi\eta rv = \frac{4}{3}\pi r^3 (\rho - \rho_0) g$$

$$v = \frac{2}{9}\frac{r^2 g(\rho - \rho_0)}{\eta}$$

Solved Numerical

Q) A steel ball of diameter d = 3.0mm starts sinking with zero initial velocity in oil whose viscosity is 0.9P. How soon after the beginning of motion will the velocity of the ball differ from the steady state velocity by n = 1.0%? Density of steel = 7.8×10^3 kg/m^3
Solution: Initial acceleration is maximum and becomes zero thus acceleration is not constant:
Viscocity = 0.9P = 0.09 kgm^{-1}s^{-1}

Net force on ball = W – F_B - F_v
F_B = Buoyant force up ward F_v = viscous force upwards , W = weight of ball down wards

Force = ma thus

$$m\frac{dv}{dt} = mg - F_B - 6\eta'\pi r v$$

Let A = mg-F_B is constant and B = $6\eta\pi r$ is another constant

$$m\frac{dv}{dt} = A - Bv$$

$$m\frac{dv}{(A - Bv)} = dt$$

Velocity after time t differs from the steady state velocity by n = 1.0%
v = (1-n)v' here v' is terminal velocity

$$m\int_0^{(1-n)v'} \frac{dv}{(A - Bv)} = \int_0^t dt$$

$$-\frac{m}{B} ln\,[\frac{A - B(1-n)v'}{A}] = t$$

At steady state net force is zero

A - Bv' = 0 ∴ v_s = A/B

$$t = -\frac{m}{B} ln\,[\frac{A - B(1-n)\frac{A}{B}}{A}]$$

$$t = -\frac{m}{B}\ln n$$

$$t = -\frac{m}{6\eta\pi r}\ln n$$

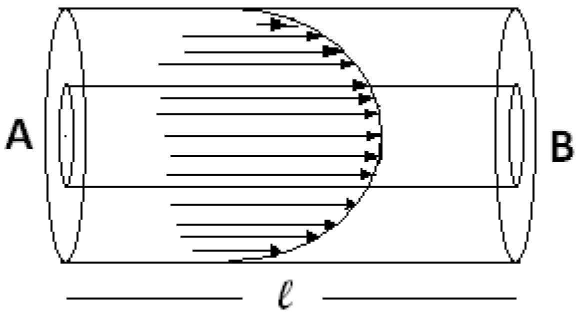
$$t = -\frac{\frac{4}{3}\pi r^3\rho}{6\eta\pi r}\ln n$$

$$t = -\frac{2r^2\rho}{9\eta}\ln n$$

$$t = -\frac{2\left(\frac{3\times10^{-3}}{2}\right)^2 \left(7.8\times10^{3}\right)}{9(0.09)}\ln(0.01)$$

t = 0.2 sec

Q) As shown in figure laminar flow is obtained in a tube of internal radius r and length l. To maintain such flow, the force balancing the viscous force obtained by producing the pressure difference(P) across the ends of the tube. Derive the equation of velocity of a layer situated at distance 'x' from the axis of the tube



Solution

Consider a cylindrical layer of radius x as shown in figure. The force acting on it are as follows

(1) At face A let pressure be P_1 Thus force $F_1 = \pi x^2 P_1$
(2) At face B let pressure be P_2 ($<P_1$) Thus force $F_2 = \pi x^2 P_2$ is against F_1
(3) Viscous force $F_3 = \eta A \left(-\frac{dv}{dx}\right)$

A is curved area of cylinder of radius x, thus A = 2πxl

Negative sign indicates as we go from axis of cylinder to walls of cylinder velocity decreases

Viscous force F_3

$$F_3 = -\eta(2\pi xl)\frac{dv}{dx}$$

For the motion of the cylinder layer with a constant velocity

$F_3 = F_1 - F_2$

$$-\eta(2\pi xl)\frac{dv}{dx} = \pi x^2 P_1 - \pi x^2 P_2$$

$$-\eta(2\pi xl)\frac{dv}{dx} = \pi x^2 (P_1 - P_2)$$

$$-\eta(2\pi xl)\frac{dv}{dx} = \pi x^2 (P)\ [\because P_1 - P_2 = P]$$

$$-dv = \frac{P}{2\eta l}x\,dx$$

At x = r, v = 0 and at x = x , v = v , v so integrating the above equation in these limits we get

$$-\int_v^0 dv = \int_x^v \frac{P}{2\eta l}x\,dx$$

$$-[v]_v^0 = \frac{P}{4\eta l}[x^2]_x^r$$

$$-[0 - v] = \frac{P}{4\eta l}[r^2 - x^2]$$

$$v = \frac{P}{4\eta l}(r^2 - x^2)$$

If we want to fing the volume of liquid flowing the tube in one second
Then velocity at axis x= 0

$$v = \frac{Pr^2}{4\eta l}$$

At the wall (x = r) velocity is zero
∴ Average velocity

$$< v > = \frac{Pr^2}{8\eta l}$$

Now volume of liquid = (average velocity)(Area of cross-section)

$$V = \frac{Pr^2}{8\eta l}(\pi r^2)$$

$$V = \frac{P\pi r^2}{8\eta l}$$

Above equation is called Poiseiulle's Law

KINETIC THEORY OF GASES AND THERMODYNAMICS

SECTION I

Kinetic theory of gases

Some important terms in kinetic theory of gases

Macroscopic quantities:

Physical quantities like pressure, temperature, volume, internal energy are associated with gases. These quantities are obtained as an average combined effect of the process taking place at the microscopic level in a system known as macroscopic quantities. These quantities can be directly measured or calculated with help of other measurable macroscopic quantities

Macroscopic description:

The description of a system and events associated with it in context to its macroscopic quantities are known as macroscopic description.

Microscopic quantities:

Physical quantities like speed, momentum, kinetic energy etc. associated with the constituent particle at microscopic level, are known as microscopic quantities

Microscopic description:

When the system and events associated with it are described in context to microscopic quantities, this description is known as microscopic description

Postulates of Kinetic theory of gases

(1) A gas consists of a very large number of molecules. Each one is a perfectly identical elastic sphere.

(2) The molecules of a gas are in a state of continuous and random motion. They move in all directions with all possible velocities.

(3) The size of each molecule is very small as compared to the distance between them. Hence, the volume occupied by the molecule is negligible in comparison to the volume of the gas.

(4) There is no force of attraction or repulsion between the molecules and the walls of the container.

(5) The collisions of the molecules among themselves and with the walls of the container are perfectly elastic. Therefore, momentum and kinetic energy of the molecules are conserved during collisions.

(6) A molecule moves along a straight line between two successive collisions and the average distance travelled between two successive collisions is called the mean free path of the molecules.
(7) The collisions are almost instantaneous (i.e) the time of collision of two molecules is negligible as compared to the time interval between two successive collisions.

Behavior of gases

It has been observed from experiments that for very low densities, the pressure, volume and temperature of gas are interrelated by some simple relations.

Boyle's law

At constant temperature and low enough density, the pressure of a given quantity (mass) of gas is inversely proportional to its volume
Thus at constant mass and constant temperature

$$P \propto \frac{1}{V}$$

Or PV = Constant

Charles's law

At constant pressure and low enough density, the volume of a given quantity (mass) of a gas is proportional to its absolute temperature
Thus at constant mass and constant pressure

$$V \propto T$$

Or $\frac{V}{T} = constant$

Gay Lussac's law

For a given volume and low enough density the pressure of a given quantity of gas is proportional to its absolute temperature.
Thus at constant mass and constant volume

$$P \propto T$$

Or $\frac{P}{T} = constant$

Avogadro's Number

"For given constant temperature and pressure, the number of molecules per unit volume is the same for all gases"
At standard temperature (273K) and pressure (1 atm), the mass of 22.4 litres of **any** gas is equal to its molecular mass (in grams). This quantity of gas is called 1 mole.
The number of particles (atoms or molecules) in one mole of substance (gas) is called Aveogadro number, which has a magnitude $N_A = 6.023 \times 10^{23}$ mol^{-1}

If N is the number of gas molecules in a container , then the number of mole of given gas is

$$\mu = \frac{N}{N_A}$$

If M is the total mass of gas in a container, and mass of one mole of gas called molar mass M_O, then the number of moles of gas is

$$\mu = \frac{M}{M_O}$$

Other important laws of an ideal gas

Grahm's law of diffusion states that when two gases at the same pressure and temperature are allowed to diffuse into each other, the rate of diffusion of each gas is inversely proportional to the square root of the density of the gas

$$rate\ of\ diffusion \propto \sqrt{\frac{1}{density\ of\ gas}}$$

Dalton's law of partial pressure states that the pressure exerted by a mixture of several gases equals the sum of the pressure exerted by each gas occupying the same volume as that of the mixture

$P_1, P_2, \ldots, P_n$ are the pressure exerted by individual gases of the mixture, then pressure of the mixture of the gas is

$P = P_1 + P_2 + \ldots + P_n$

Ideal gas-state equation and it different forms

If we combine Boyle's law and Charle's law we get

$$\frac{PV}{T} = constant$$

For a given quantity of gas, which shows that for constant temperature and pressure, if quantity or mass of gas is varies, then volume of the gas is proportional to the quantity of gas.

Thus constant on the right hand side of the equation depends on the quantity of the gas. If quantity is represented in mole then

$$\frac{PV}{T} = \mu R$$

Equation is called an ideal gas-state equation

Here R is universal gas constant = 8.314 J $mole^{-1}$ K^{-1}

If gas completely obeys equation

$$PV = \mu RT \quad --- eq(1)$$

at all values of pressure and temperature, then such a (imaginary) gas is called an ideal gas. By putting

$$\mu = \frac{N}{N_A}$$

In above equation we get

$$PV = \frac{N}{N_A} RT = N \frac{R}{N_A} T$$

Putting $R/N_A = k_B$ (Boltzmann's constant = 1.38×10^{-23} J K^{-1})

$$PV = Nk_BT$$

$$\therefore P = \frac{N}{V} k_BT$$

If n = N/V number of molecules per unit volume of gas

$$\therefore P = nk_BT \quad --- eq(2)$$

Putting

$$\mu = \frac{M}{M_0}$$

In equation (1)

$$PV = \frac{M}{M_0} RT$$

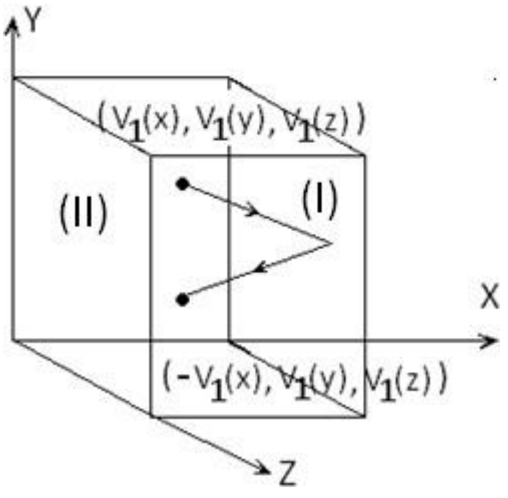
$$P = \frac{M}{V} \frac{RT}{M_0}$$

$$P = \frac{\rho RT}{M_0} \quad --- eq(3)$$

ρ is the density of gas

Pressure of an ideal gas and rms speed of gas molecules

The molecules of a gas are in a state of random motion. They continuously collide against the walls of the container. During each collision, momentum is transferred to the walls of the container.



The pressure exerted by the gas is due to the continuous collision of the molecules against the walls of the container. Due to this continuous collision, the walls experience a continuous force which is equal to the total momentum imparted to the walls per second. The force experienced per unit area of the walls of the container determines the pressure exerted by the gas.

Consider a cubic container of side L containing n molecules of perfect gas moving with velocities C_1, C_2, C_3 ... C_n

A molecule moving with a velocity v_1, will have velocities $C_1(x)$, $C_1(y)$ and $C_1(z)$ as components along the x, y and z axes respectively. Similarly $C_2(x)$, $C_2(y)$ and $C_2(z)$ are the velocity components of the second molecule and so on.

Let a molecule P shown in figure having velocity C_1 collide against the wall marked I perpendicular to the x-axis. Only the x-component of the velocity of the molecule is relevant for the wall . Hence momentum of the molecule before collision is *m* $C_1(x)$ where *m* is the mass of the molecule.
Since the collision is elastic, the molecule will rebound with the velocity $C_1(x)$ in the opposite direction. Hence momentum of the molecule after collision is $-mC_1(x)$

Change in the momentum of the molecule = Final momentum - Initial momentum
Change in the momentum of the molecule = - $mC_1(x)$ - $mC_1(x)$= $-2mC_1(x)$
During each successive collision on face I the molecule must travel a distance 2L from face I to face II and back to face I.

Time taken between two successive collisions is = $2L/ C_1(x)$

$$\therefore Rate\ of\ change\ of\ momentum = \frac{change\ in\ momentum}{time\ taken}$$

$$Rate\ of\ change\ of\ momentum = \frac{-2mC_1(x)}{\frac{2L}{V_1(x)}} = \frac{-mC_1^2(x)}{L}$$

$$\text{Force exerted on the molecule} = \frac{-mC_1^2(x)}{L}$$

According to Newton's third law of motion, the force exerted by the molecule =

$$= -\frac{-mC_1^2(x)}{L} = \frac{mC_1^2(x)}{L}$$

Force exerted by all the *n* molecules is

$$F_x = \frac{mC_1^2(x)}{L} + \frac{mC_2^2(x)}{L} + \frac{mC_3^2(x)}{L} + \cdots + \frac{mC_n^2(x)}{L}$$

Pressure exerted by the molecules

$$P_x = \frac{F_x}{A}$$

$$P_x = \frac{1}{L^2}\left(\frac{mC_1^2(x)}{L} + \frac{mC_2^2(x)}{L} + \frac{mC_3^2(x)}{L} + \cdots + \frac{mC_n^2(x)}{L}\right)$$

$$P_x = \frac{m}{L^3}\left(C_1^2(x) + C_2^2(x) + C_3^2(x) + \cdots + C_n^2(x)\right)$$

Similarly, pressure exerted by the molecules along Y and Z axes are

$$P_y = \frac{m}{L^3}(C_1^2(y) + C_2^2(y) + C_3^2(y) + \cdots + C_n^2(y))$$

$$P_z = \frac{m}{L^3}(C_1^2(z) + C_2^2(z) + C_3^2(z) + \cdots + C_n^2(z))$$

Since the gas exerts the same pressure on all the walls of the container

$$P_x = P_y = P_z$$

$$P = \frac{P_x + P_y + P_z}{3}$$

$$P = \frac{1}{3}\frac{m}{L^3}[(C_1^2(x) + C_2^2(x) + C_3^2(x) + \cdots + C_n^2(x)) + (C_1^2(y) + C_2^2(y) + C_3^2(y) + \cdots + C_n^2(y)) + (C_1^2(z) + C_2^2(z) + C_3^2(z) + \cdots + C_n^2(z))]$$

$$P = \frac{1}{3}\frac{m}{L^3}[(C_1^2(x) + C_1^2(y) + C_1^2(z)) + (C_2^2(x) + C_2^2(y) + C_2^2(z)) + \cdots + (C_n^2(x) + C_n^2(y) + C_n^2(z))]$$

$$P = \frac{m}{3V}[C_1^2 + C_2^2 + .. + C_n^2]$$

$$P = \frac{mn}{3V}\left[\frac{C_1^2 + C_2^2 + .. + C_n^2}{n}\right]$$

$$P = \frac{mn}{3V} < C^2 >$$

Here V is volume of gas

Where $< C^2 >$ is called the root mean square (RMS) velocity, which is defined as the square root of the mean value of the squares of velocities of individual molecules.

Since mn = mass of gas and density ρ = mass/volume

$$P = \frac{\rho}{3} < C^2 >$$

Relation between the pressure exerted by a gas and the mean kinetic energy of translation per unit volume of the gas

Mean kinetic energy of translation per unit volume of the gas

$$E = \frac{1}{2}\rho < C^2 >$$

Thus

$$\frac{P}{E} = \frac{\frac{\rho}{3} < C^2 >}{\frac{1}{2}\rho < C^2 >} = \frac{2}{3}$$

Or P = (2/3)E

Average kinetic energy per molecule of the gas

Let us consider one mole of gas of mass M and volume V.

$$P = \frac{\rho}{3} < C^2 >$$

$$P = \frac{M}{3V} < C^2 >$$

$$PV = \frac{M}{3} < C^2 >$$

From ideal gas equation for one mole of gas

PV = RT

$$\frac{M}{3} < C^2 > = RT$$

$$M < C^2 > = 3RT$$

$$\frac{1}{2} M < C^2 > = \frac{3}{2} RT$$

Average kinetic energy of one mole of the gas is equal to = (3/2) RT

Since one mole of the gas contains N_A number of atoms where N_A is the Avogadro number we have $M = N_A m$

$$\frac{1}{2} N_A m < C^2 > = \frac{3}{2} RT$$

$$\frac{1}{2} m < C^2 > = \frac{3}{2}\frac{R}{N_A} T$$

$$\frac{1}{2} m < C^2 > = \frac{3}{2} k_B T$$

k_B is Boltzmann constant

Average kinetic energy per molecule of the gas is equal to (3/2) k_BT

Hence, it is clear that the temperature of a gas is the measure of the mean translational kinetic energy per molecule of the gas

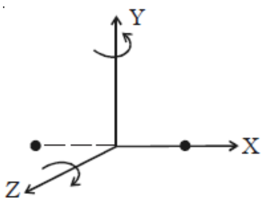
Degrees of freedom

The number of degrees of freedom of a dynamical system is defined as the total number of co-ordinates or independent variables required to describe the position and configuration of the system.

(i) A particle moving in a straight line along any one of the axes has one degree of freedom (e.g) Bob of an oscillating simple pendulum.
(ii) A particle moving in a plane (X and Y axes) has two degrees of freedom. (eg) An ant that moves on a floor.
(iii) A particle moving in space (X, Y and Z axes) has three degrees of freedom. (eg) a bird that flies.
A point mass cannot undergo rotation, but only translatory motion. Three degree of freedom
A rigid body with finite mass has both rotatory and translatory motion.
The rotatory motion also can have three co-ordinates in space, like translatory motion ; Therefore a rigid body will have six degrees of freedom ; three due to translatory motion and three due to rotator motion.

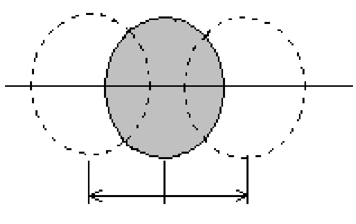
Monoatomic molecule
Since a monoatomic molecule consists of only a single atom of point mass it has three degrees of freedom of translatory motion along the three co-ordinate axes
Examples : molecules of rare gases like helium, argon, etc.

Diatomic molecule rigid rotator



The diatomic molecule can rotate about any axis at right angles to its own axis. Hence it
has two degrees of freedom of rotational motion in addition to three degrees of freedom of translational motion along the three axes. So, a diatomic molecule has five degrees of freedom (Fig.). Examples : molecules of O_2, N_2, Cl_2, etc

Diatomic molecule like CO : Have five freedom as stated in rigid rotator apart from that they have two more freedoms due to vibration (oscillation) about mean position



Plyatomic molecules posses rotational kinetic energy energy of vibration in addition to their translational energy.
Therefore when heat energy is given to such gases, it is utilized in increasing the translational kinetic energy, rotational kinetic energy and vibrational kinetic energy of the

gas molecules and hence more heat is required. This way polyatomic molecules posses more specific heat

Law of equipartition of energy

Law of equipartition of energy states that for a dynamical system in thermal equilibrium the total energy of the system is shared equally by all the degrees of freedom. The energy associated with each degree of freedom per moelcule is (1/2)kT
where *k* is the Boltzmann's constant.
Let us consider one mole of a monoatomic gas in thermal equilibrium at temperature T.
Each molecule has 3 degrees of freedom due to translatory motion.
According to kinetic theory of gases, the mean kinetic energy of a molecule is (3/2)kT
Let *Cx* , *Cy* and *Cz* be the components of RMS velocity of a molecule along the three axes.
Then the average energy of a gas molecule is given by

$$\frac{1}{2}mC^2 = \frac{1}{2}mC_x^2 + \frac{1}{2}mC_y^2 + \frac{1}{2}mC_z^2$$

$$\frac{1}{2}mC_x^2 + \frac{1}{2}mC_y^2 + \frac{1}{2}mC_z^2 = \frac{3}{2}kT$$

Since molecules move at random, the average kinetic energy corresponding to each degree of freedom is the same.

$$\frac{1}{2}mC_x^2 = \frac{1}{2}mC_y^2 = \frac{1}{2}mC_z^2 = \frac{1}{2}kT$$

∴ Mean kinetic energy per molecule per degree of freedom is (1/2) kT

Mean free path

"The linear distance travelled by a molecule of gas with constant speed between two consecutive collisions (between molecules) is called free path. The average of such free paths travelled by a molecule is called mean free path"
Suppose molecules of gas are spheres of diameter d. If the centre between the two molecules is less or equal to d then they will collide when they come close.
Consider a molecule of diameter d moving with average speed v, and the other molecule is stationary. The molecule under consideration will sweep a cylinder of $\pi d^2 vt$. In time t.
If the number of molecules per unit volume is n, then the number of molecules in this cylinder is $n\pi d^2 vt$. Hence the molecule will under go $n\pi d^2 vt$ collisions in time t
The mean free path l is the average distance between two successive collision

$$Mean\ free\ path = \frac{distance\ travelled\ in\ time\ t}{number\ of\ collisions\ in\ time\ t}$$

$$l = \frac{vt}{2\pi d^2 vt}$$

$$l = \frac{1}{2\pi n d^2}$$

In this derivation other molecules are considered stationary. In actual practice all gas molecules are moving and there collision rate is determined by the average relative velocity < V > Hence mean free path formula is $l = \frac{1}{\sqrt{2}\pi \bar{n} d^2}$

Solved Numerical

Q) Find the mean translational kinetic energy of a molecules of He at 27^{O}

Solution: Since He is mono atomic degree of freedom is 3

Kinetic energy =(3/2)k_BT

Here k_B = Boltzmann's constant = 1.38×10^{-23} J K^{-1} Temperature T = 27+273 = 300 K

$$K = \frac{3}{2} \times 1.38 \times 10^{-23} \times 300 = 6.21 \times 10^{-21}\ J$$

Q) At what temperature rms velocity of O_2 is equal to rms velocity of H_2 ay 27^{O}C?

Solution

Kinetic energy

$$\frac{1}{2} m < C^2 > = \frac{3}{2} k_B T$$

$$< C^2 > = \frac{3k_BT}{m}$$

But rms velocity of O_2 rms velocity of He

$$\frac{3k_BT}{32} = \frac{3k_B \times 300}{4}$$

$$T = 2400\ K$$

Q) Find rms velocity of hydrogen at 0^{O}C temperature and 1 atm pressure. Density of hydrogen gas is 8.9×10^{-2} kg m^{-3}

Solution:

From formula

$$P = \frac{\rho}{3} < C^2 >$$

$$< C > = \sqrt{\frac{3P}{\rho}}$$

$$< C > = \sqrt{\frac{3 \times 1.01 \times 10^5}{8.9 \times 10^{-2}}} = 1845\ ms^{-1}$$

Q) If the molecular radius of hydrogen molecule is 0.5Å, find the mean free path of hydrogen molecule at 0^{O}C temperature and 1 atm pressure

Solution

d = 2×r = 1 Å

From formula P = nK_BT

$$n = \frac{P}{k_BT}$$

$$n = \frac{1.01 \times 10^{5}}{1.38 \times 10^{-23} \times 273} = 2.68 \times 10^{25}$$

From formula for mean path $l = \frac{1}{\sqrt{2} \times 3.14 \times 2.68 \times 10^{25} \times (1 \times 10^{-10})^{2}} = 8.4 \times 10^{-7}\ m$

SECTION II
Thermodynamics

Some important terms

Thermodynamic system : It is apart of the universe under thermodynamic study. A system can be one, two or three dimensional. May consists of single or many objects or radiation

Environment : remaining part of universe around the thermodynamic system is Environment. Environment have direct impact on the behavior of the system

Wall: The boundary separating the Stem from the universe is wall

Thermodynamic co-ordinates: The macroscopic quantities having direct effect on the internal state of the system are called thermodynamic coordinates. For example
Take the simple example of a sample of gas with a fixed number of molecules. It need not be ideal. Its temperature, *T*, can be expressed as a function of just two variables, volume, *V*, and pressure, *p*. We can, it turns out, express all gas properties as functions of just two variables (such as *p* and *V* or *p* and *T*). These properties include refractive index, viscosity, internal energy, entropy, enthalpy, the Helmholtz function, the Gibbs function. We call these properties 'functions of state'. The state is determined by the values of just two variables

Thermodynamic system: The system represented by the thermodynamic co-ordinate is called a thermodynamic system

Thermodynamic process: The interaction between a system and its environment is called a thermodynamic process

Isolated system: If a system does not interact with its surrounding then it is called an isolated system. Thermal and mechanical properties of such system is said to be ina definite thermodynamic equilibrium state
Heat (Q) and Work(W): The amount of heat energy exchanged during the interaction of system with environment is called heat (Q) and the mechanical energy exchanged is called work (W).

Thermodynamic variables: Thermodynamic variables describe the momentary condition of a thermodynamic system. Regardless of the path by which a system goes from one state to another — i.e., the sequence of intermediate states — the total changes in any state variable will be the same. This means that the incremental changes in such variables are exact differentials. Examples of state variables include: Density (ρ), Energy (E), Gibbs free energy (G), Enthalpy (H) , Internal energy (U), Mass (m) , Pressure (p) ,Entropy (S) Temperature (T), Volume (V)

Extensive thermodynamic state variable: The variables depending on the dimensions of the system are called extensive variables. For examples mass, volume, internal energy

Intensive thermodynamic state variable: The variables independent on the dimensions of the system are called intensive variables. For examples pressure, temperature, density

Thermal equilibrium: When two system having different temperatures are brought in thermal contact with each other, the heat flows from the system at higher temperature to that at lower temperature. When both the system attains equal temperatures, the net heat exchanged between them becomes zero. In this state they are said to be in thermal equilibrium state with each other.

Zeroth Law of thermodynamics: "If the system A and B are in the thermal equilibrium with a third system C, then A and B are also in thermal equilibrium with each other"

Temperature may be defined as the particular property which determines whether a system is in thermal equilibrium or not with its neighbouring system when they are brought into contact

adiabatic wall – an insulating wall (can be movable) that does not allow flow of energy (heat) from one to another**.**

diathermic wall – a conducting wall that allows energy flow (heat) from one to another

Specific heat capacity

Specific heat capacity of a substance is defined as the quantity of heat required to raise the temperature of 1 kg of the substance through 1K. Its unit is J $kg^{-1}K^{-1}$.

Molar specific heat capacity of a gas

Molar specific heat capacity of a gas is defined as the quantity of heat required to raise the temperature of 1 mole of the gas through 1K. Its unit is J mol^{-1} K^{-1}.

Let *m* be the mass of a gas and *C* its specific heat capacity. Then $\Delta Q = m \times C \times \Delta T$ where ΔQ is the amount of heat absorbed and ΔT is the corresponding rise in temperature.

Case (i)

If the gas is insulated from its surroundings and is suddenly compressed, it will be heated up and there is rise in temperature, even though no heat is supplied from outside
(i.e) $\Delta Q = 0 \therefore C = 0$

Case (ii)

If the gas is allowed to expand slowly, in order to keep the temperature constant, an amount of heat ΔQ is supplied from outside, then

$$C = \frac{\Delta Q}{m\Delta T} = \frac{\Delta Q}{0} = +\infty$$

($\because$ ΔQ is +ve as heat is supplied from outside)

Case (iii)

If the gas is compressed gradually and the heat generated ΔQ is conducted away so that temperature remains constant, then

$$C = \frac{-\Delta Q}{m\Delta T} = \frac{-\Delta Q}{0} = +\infty$$

($\because$ ΔQ is -ve as heat is supplied by the system)

Thus we find that if the external conditions are not controlled, the value of the specific heat capacity of a gas may vary from +∞ to -∞

Hence, *in order to find the value of specific heat capacity of a gas, either the pressure or the volume of the gas should be kept constant.* Consequently a gas has two specific heat capacities

(i) Specific heat capacity at constant volume
(ii) Specific heat capacity at constant pressure.

Molar specific heat capacity of a gas at constant volume

Molar specific heat capacity of a gas at constant volume C_V is defined as the quantity of heat required to raise the temperature of one mole of a gas through 1 K, keeping its volume constant

Molar specific heat capacity of a gas at constant pressure

Molar specific heat capacity of a gas at constant pressure C_p is defined as the quantity of heat to raise the temperature of one mole of a gas through 1 K keeping its pressure constant

Specific heat of gas from the law of equipartition of energy

The energy associated with each degree of freedom is (1/2)K_BT. It means that, if the degree of freedom of a gas molecule is f then the average heat energy of each molecule of gas is

$$E_{ave} = f \times \frac{1}{2} k_B T$$

If number of moles of an ideal gas is μ , then the number of moles in the gas is μN_A.
Therefore the **internal energy** of μ mole of ideal gas is
$U = \mu N_A E_{average}$

$$U = \mu N_A f \times \frac{1}{2} k_B T$$

$$U = \frac{f}{2} \mu RT \quad (\because R = N_A k_B)$$

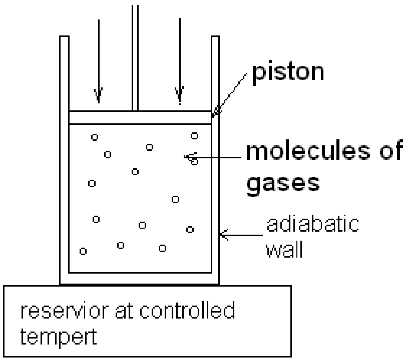
Work in thermodynamics

The amount of mechanical energy exchanged between two bodies during mechanical interaction is called work. Thus work is a quantity related to mechanical interaction. A system can possess mechanical energy, but cannot posses work

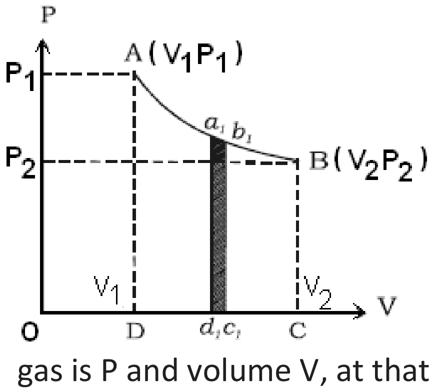
In thermodynamics the work done by the system is considered positive and the work done on the system is considered negative.

The reason behind such sign convention is due to the mode of working of heat engine in which the engine absorbs heat from the environment and converts it into work W means the energy of the system is reduced by W

Formula for the work done during the compression of gas at constant temperature



As shown in figure µ molecules of gas are enclosed in a cylindrical container at low pressure, and an air tight piston capable of moving without friction with area A is provide.. the conducting bottom of the cylinder is placed on an arrangement whose temperature can be contolled.



At constant temperature , measuring the volume of the gas for different values of pressure, the graph of P-V can be plotted as shown in figure. These types of process are called isothermal process and curved of P-V is called isotherm.

Suppose initial pressure and volume of the pas is represented by P_1 and V_1 respectively. Keeping the temperature T of the gas to be constant , volume of gas decreases slowly by pushing piston down. Let final pressure and volume of the gas be is P_2 and V_2

During the process, at one moment when pressure of the gas is P and volume V, at that time, let the piston moves inward by Δx. then the volume of the gas decreases by ΔV. this displacement is to small that there is no apparent change in pressure.

Hence work done on the gas

$\Delta W = F\Delta x$

$\Delta W = PA\Delta x$

$\Delta W = P\Delta V$
If the volume of the gas is decreasing from V_1 to V_2 through such small changes, then the total work done on the gas

$$W = \sum_{V_1}^{V_2} P \Delta V$$

If this summation is taking the limit as $\Delta V \rightarrow 0$ the summation results in integration

$$W = \int_{V_1}^{V_2} P dV$$

But the ideal gas equation for μ moles of gas is PV=μRT thus

$$W = \int_{V_1}^{V_2} \frac{\mu RT}{V} dV$$

$$W = \mu RT \int_{V_1}^{V_2} \frac{dV}{V}$$

$$W = \mu RT [\ln V]_{V_1}^{V2}$$

$$W = \mu RT [V_2 - V_1]$$

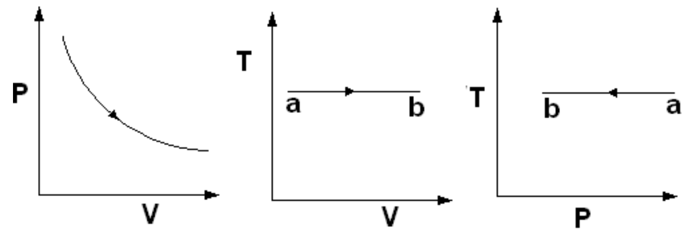
$$W = \mu RT \ln \left(\frac{V_2}{V_1}\right)$$

$$W = 2.303 \mu RT \log_{10} \left(\frac{V_2}{V_1}\right)$$

Equation does not give the work W by an ideal gas during every thermodynamic process, but it gives the work done only for a process in which the temperature is held constant.

Since $V_2 < V_1$ hence $\log(V_2/V_1)$ is negative . Thus we get negative value of work which represents that **during the compression of gas at constant temperature, the work is done on the gas**

If the gas is expanded then Since $V_2 > V_1$ hence $\log(V_2/V_1)$ is positive . Thus we get negative value of work which represents that **during the compression of gas at constant temperature, the work is done by the gas**
The P-V, T-V and T-P diagram for isothermal process will be like the curves given below



ISOTHERMAL PROCESS

Work done at constant volume and constant pressure

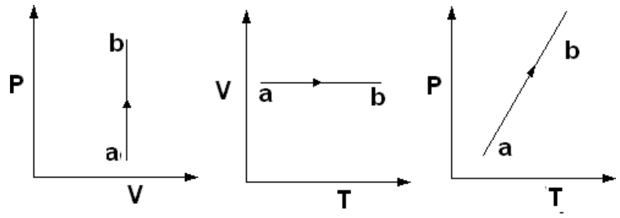
Constant volume : Also called as isochoric process

If the volume is constant then dV = from equation

$$W = \int_{V_1}^{V_2} PdV$$

Work done is zero

The P-V, V-T and P-T diagrams for isochoric process will be like curves given below



ISOCHORIC PROCES

Constant pressure :Also called as Isobaric process

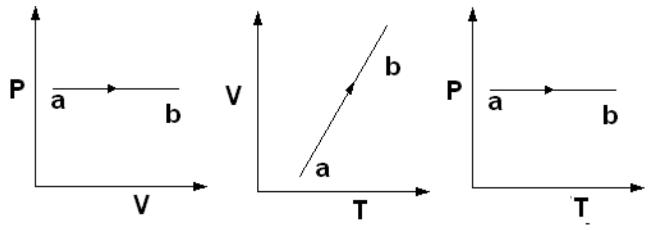
If the volume is changing while pressure is constant then from equation

$$W = \int_{V_1}^{V_2} PdV = P\int_{V_1}^{V_2} PdV$$

$$W = P[V_2 - V_1]$$

W = PΔV (for constant pressure)

The P-V, V-T and P-T diagrams for isobaric process will be like curves given below



ISOBARIC PROCESS

Work done during adiabatic process

No excahge of heat takes palce between system and it environment in this process. This is possible when (1) walls of a system are thermal insulator or (2) process is very rapid.

The relation between pressure and volume for ideal gas is

$$PV^{\gamma} = constant$$

Where $\gamma = \frac{CP}{C_V}$

For an adiabatic process

$$W = \int_{V_1}^{V_2} PdV$$

Let

$$PV^{\gamma} = A$$

$$W = \int_{V_1}^{V_2} \frac{A}{V^{\gamma}} dV$$

$$W = A \int_{V_1}^{V_2} \frac{dV}{V^{\gamma}}$$

$$W = A \left[\frac{V^{-\gamma+1}}{-\gamma+1}\right]_{V_1}^{V_2}$$

$$W = \frac{1}{1-\gamma}\left[AV_2^{-\gamma+1} - AV_1^{-\gamma+1}\right]$$

$But\ A = P_2V_2^{\gamma} = P_1V_1^{\gamma}$

$$W = \frac{1}{1-\gamma}\left[P_2V_2^{\gamma}V_2^{-\gamma+1} - P_1V_1^{\gamma}V_1^{-\gamma+1}\right]$$

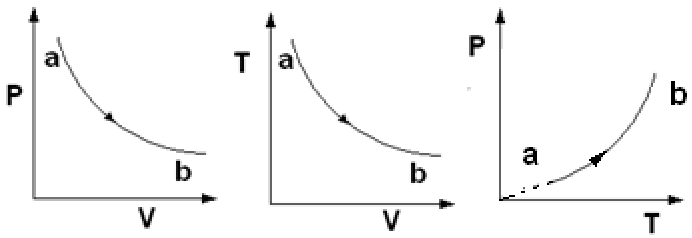
$$W = \frac{1}{1-\gamma}\left[P_2V_2 - P_1V_1\right]$$

$$W = \frac{1}{\gamma-1}\left[P_1V_1 - P_2V_2\right]$$

From ideal gas equation PV = μRT

$$W = \frac{\mu R}{\gamma - 1}\left[T_1 - T_2\right]$$

The P-V, T-V and P-T diagrams for adiabatic process will be lie the curves given below



ADIABATIC PROCESS

Solved Numerical

Q) Calculate work done if one mole of ideal gas is compressed isothermally at a temperature 27^{O} C from volume of 5 litres to 1 litre

Solution:

Formula for work done during iso-thermal process is

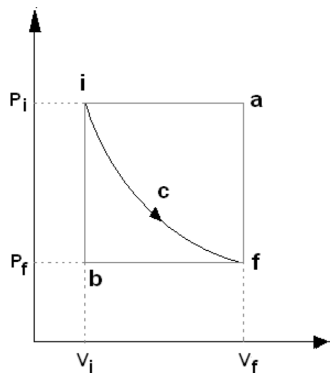
$$W = 2.303\mu RT \log_{10}\left(\frac{V_2}{V_1}\right)$$

$$W = 2.303 \times 1 \times 8.31 \times 300 \times log\left(\frac{1}{5}\right)$$

$$W = 2.303 \times 1 \times 8.31 \times 300 \times [log1 - log5]$$
$$W = 2.303 \times 1 \times 8.31 \times 300 \times [0 - 0.6990]$$

W = -4012.5J

First law of thermodynamics



Suppose a sytem absorbs heat and as a result work is done by it (by the system). We can think of different paths (process) through which the system can be taken from initial stage (i) to final state (f)

For the process iaf, ibf, icf. Suppose the heat absorbed by the system are Q_a, Q_b , Q_c respectively and the values of the work done are respectively W_a, W_b, W_c. Here

$Q_a \neq Q_b \neq Q_c$ and $W_a \neq W_b \neq W_c$, but difference of heat and work done turns out to be same

$Q_a - W_a = Q_b - W_b = Q_c - W_c$

Thus value of Q – W depends only on initial and final state of the system. A thermodynamic state function can be defined such that the difference between any two states is equal to Q – W. Such a function is called internal energy U of system

The system gains energy Q in the form of heat energy and spends energy W to do work. Hence the internal energy of the system changes by Q-W.

If the internal energies of system in initial state is U_i and final state is U_f then

$U_i - U_f = \Delta U = Q - W$ Which is the first law of thermodynamics

The first law is obeyed in all the changes occurring in nature

Isochoric process

Since in this process volume remains constant, the work done in this process is equal to zero. Applying first law of thermodynamics to this process, we get

$\Delta Q = \Delta U + \Delta W$

$\Delta Q = \Delta W$

So heat exchange in this process takes place at the expense of the internal energy of the system.

dQ = dU

$$\left(\frac{dQ}{dT}\right)_V = \left(\frac{dU}{dT}\right)_V$$

$$since\ U = \frac{f}{2}RT$$

$$\left(\frac{dQ}{dT}\right)_V = \frac{f}{2}R$$

Thus above equation is for the energy required to increase temperature by one unit of one of ideal gas it is molar specific heat at constant volume C_V

$$\left(\frac{dQ}{dT}\right)_V = \frac{f}{2}R = C_V$$

Isobaric process

Applying first law of thermodynamics to isobaric process we get

$$\Delta Q = \Delta U + P(V_2 - V_1)$$

$$\Delta Q = \Delta U + P\Delta V$$

But PV = RT for one mole of gas

$$\therefore P\Delta V = R\Delta T \text{ thus}$$

$$\therefore \Delta Q = \Delta U + R\Delta T$$

$$since\ U = \frac{f}{2}RT$$

$$\therefore \left(\frac{dQ}{dT}\right)_P = \frac{f}{2}R\frac{dT}{dT} + \mathrm{R}\frac{dT}{dT}$$

$$\therefore \left(\frac{dQ}{dT}\right)_P = \frac{f}{2}R + \mathrm{R}$$

Since dQ/dT is specific heat at constant pressure = C_P

$$\therefore C_P = C_V + R$$

$$\text{OR } C_P - C_V = R$$

$$\gamma = \frac{C_P}{C_V} = \frac{\frac{f}{2}R + \mathrm{R}}{\frac{f}{2}R}$$

$$\gamma = 1 + \frac{2}{f}$$

f is degree of freedom

For monoatomic molecule f = 3

$$C_V = \frac{3R}{2}, C_P = \frac{5R}{2}, \gamma = \frac{5}{3}$$

For the diatomic molecules (rigid rotator) f = 5

$$C_V = \frac{5R}{2}, C_P = \frac{7R}{2}, \gamma = \frac{7}{5}$$

For the diatomic molecules (with vibration , molecule like CO) f = 7

$$C_V = \frac{7R}{2}, C_P = \frac{9R}{2}, \gamma = \frac{9}{7}$$

According to the equipartion theorem the change in internal energy is related to the temperature of the system by

$$\Delta U = mC_V\Delta T$$

Isothermal process

For isothermal process ΔU = 0.

Applying first law of thermodynamics we get

ΔQ = W

$$\Delta Q = W = 2.303\mu RT\log_{10}\left(\frac{V_2}{V_1}\right)$$

Adiabatic process

Applying first law of thermodynamics we get

ΔQ = ΔU + ΔW

For adiabatic process ΔQ = 0

$$-\Delta U = \Delta W$$

The reduction in internal energy of the gas (due to which temperature fails) is equal to the work done during an adiabatic expansion. Again during an adiabatic compression the work done on the gas causes its temperature rise. Adiabatic processes are generally very fast.

Example when we use air pump to fill air in bicycle tyre, pump get heated on pumping rapidly

Solved Numerical

Q) At 27°C, two moles of an ideal momoatomic gas occupy a volume V. The gas expands adiabatically to a volume 2V. Calculate (a) final temperature of the gas (b) Change in its internal energy (c) Work done by the gas during the process

Take R = 8.31 J/mole/K

Solution:

For monoatomic gas γ = 5/3.

T = 27+273 =300

(a) Gas expanded adiabatically

$$P_2V_2^{\gamma} = P_1V_1^{\gamma}$$

Since PV∝T

P∝ T/V

Thus

$$T_2V_2^{\gamma-1} = T_1V_1^{\gamma-1}$$

$$\therefore T_2 = T_1\left(\frac{V_1}{V_2}\right)^{\gamma-1}$$

$$\therefore T_2 = 300\left(\frac{1}{2}\right)^{5/3-1} = 189K$$

(b) For adiabatic process ΔQ = 0

$$-\Delta U = \Delta W$$

$$W = -\Delta U = \frac{\mu R}{\gamma - 1}[T_1 - T_2]$$

$$-\Delta U = \frac{2 \times 8.31}{\frac{5}{3} - 1}[300 - 189] = 2767.23\,J$$

ΔU = -2767.23 J

(c) ΔW = -ΔU

ΔW = 2767.23 J

Isothermal and adiabatic curves

The relation between the pressure and volume of gas can be represented graphically. The curve for an isothermal process is called isothermal curve or an isotherm and there are different isotherms for different temperatures for a given gas. A similar curve for an adiabatic process is called an adiabatic curve or adiabatic

Since

$$\left(\frac{dP}{dV}\right)_{isothermal} = -\frac{P}{V}$$

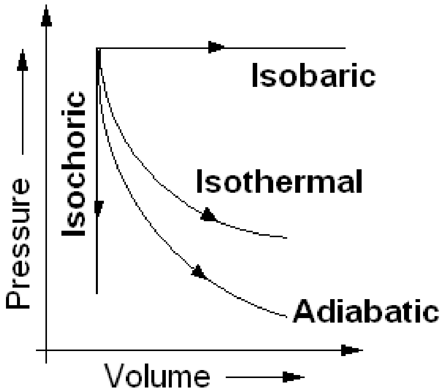
And

$$\left(\frac{dP}{dV}\right)_{adiabatic} = -\gamma\frac{P}{V}$$

So

$$\left(\frac{dP}{dV}\right)_{adiabatic} = \gamma\left(\frac{dP}{dV}\right)_{isothermal}$$

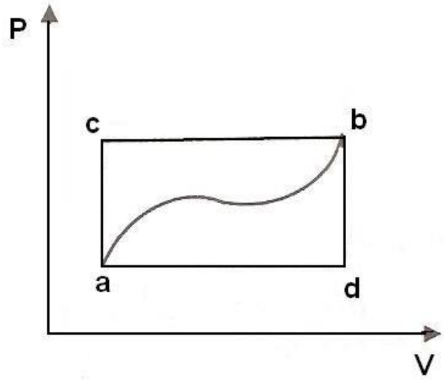
Since γ > 1, so adiabatic curve is steeper-than the isothermal curve



To permit comparison between isothermal ,adiabatic process, Isochoric and isobaric process an isothermal curve ,an adiabatic curve isochoric and Isobaric curves of gas are drawn on the same pressure-volume diagram starting from the same point.

Solved Numerical

Q) When a system is taken from state a to state b along the path acb it is found that a quantity of heat Q = 200J is absorbed by the system and a work W = 80J is done by it. Along the path adb, Q = 144J



(i) What is the work done along the path adb

(ii) IF the work done on the system along the curvered path ba is 52J, does the system absorb or linerate heat and how much

(iii) If U_a = 40J, what is U_b

(iv) If U_d = 88J, what is Q for the path db and ad?

Solution

From the first law of thermodynamics, we have

$Q = \Delta U + \Delta W$

$Q = (U_b - U_a) + W$

Where U_b is the internal energy in the state b and U is the internal energy in the stste $_a$

For the path acb, it is given that

Q = 200J (absorption) and

Q = 80J (work done by the system)

$\therefore U_b - U_a = = Q - W = 200-80 = 120J$

Which is the increase in the internal energy of the system for path acb. Whatever be the path between a and b the change in the internal energy will be 120 J only

(i) To determine the work done along the path adb

Given Q = 144J

$\Delta U = U_b - U_a = 120J$

$Q = (U_b - U_a) + W$

144 = 120 + W

W = 24J

Since W is positive, work is done by the system

(ii) For the curved return path ba, it is given that

Given W=-52J (work done on the system)

ΔU = -120 J (negative sign since $\Delta U = U_a - U_b$)

$Q = (U_a - U_b) + W$

Q = (-120 – 52) J = -172 J

Negative sign indicates heat is extracted out of the system

(iii) Since $U_b - U_a$ = 120 J and U_a = 40 j

$U_b = U_a + 120 = 40 + 120 = 160J$

(iv) For path db, the process is isochoric since it is at constant volume

Work done is zero

$Q = \Delta U + W$

$Q = \Delta U$

$Q = U_b - U_d = 160 - 88 = 72$ J

For the path ad,

$Q = Q_{adb} - Q_{db} = 144J - 72\ J = 72J$

Q) A mass of 8 g of oxygen at the pressure of one atmosphere and at temperature 27^{O}C is enclosed in a cylinder fitted with a frictionless piston. The following operations are performed in the order given

(a) The gas is heated at constant pressure to 127^{O}C

(b) then it is compressed isothermally to its initial volume and

(c) finally it is cooled to its initial temperature at constant volume

(i) What is the heat absorbed by the gas during process (A)?

(ii) How much work is done by the gas in process A

(iii)What is the work done on the gas in process B

(iv)How much heat is extracted from the gas in process (c)

[Specific heat capacity of oxygen C_V = 670 J/KgK;]

Solution:

Volume of gas at temperature 27+273 = 300K = T_2

Molecular weight of Oxygen = 32 thus 8g = 0.2 mole

At STP volume of 1 mole is 22.4 litre Thus volume of 0.25 mole is V_1 = 22.4/4

Thus for formula volume at 27^{O}C is

$$\frac{V_2}{T_2} = \frac{V_1}{T_1}$$

$$V_2 = \frac{T_2}{T_1} V_1$$

$$V_2 = \frac{300}{273} \times \frac{22.4}{4} = \frac{560}{91} \times 10^{-3}\, m^3$$

Similarly

Volume at 127^{O}C is

$$V_3 = V_2 \times \frac{400}{300} = \frac{4}{3} V_2$$

$$\frac{V_3}{V_2} = \frac{4}{3}$$

(i)

For Isothermal compression

dQ = dU + dW = $mC_v\Delta T$ + $P(V_3 - V_2)$

$$dQ = \frac{8}{1000} \times 670 \times 100 + 1.013 \times 10^5 \times [\frac{560 \times 10^{-3}}{3 \times 91}]$$

dQ = 536 + 207.8 = 743.8 J

(ii) dW = $P(V_3 - V_2)$ = 207.8 J

(iii) Work done in compressing the gas isothermally =

$$W = 2.303\mu RT \log_{10} (\frac{V_3}{V_2})$$

$$W = 2.303 \times \frac{m}{M} RT \log_{10}\left(\frac{V_3}{V_2}\right)$$

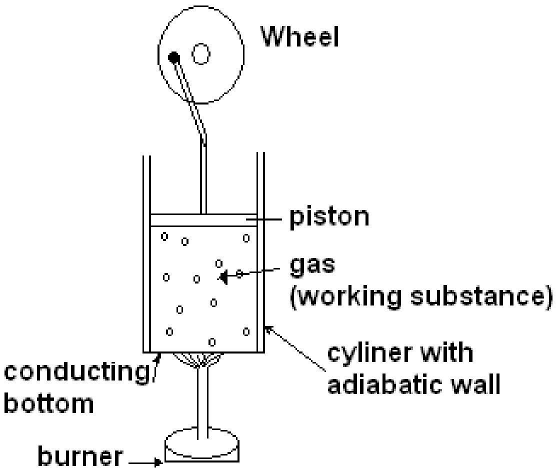
$$W = 2.303 \times \frac{8}{32} \times 8.31 \times 400 \times \log_{10}\left(\frac{4}{3}\right)$$

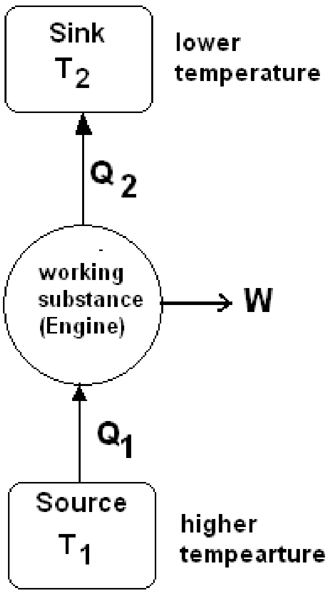
$$W = 831 \times 0.2877 = 239.1\ J$$

(d) Heat given out by the gas in stage (C) = $mC_V\Delta T$

$$\frac{8}{1000} \times 670 \times 100 = 536\ J$$

Heat Engine





A device converting heat energy into mechanical work is called heat engines.

A simple heat engine is shown in figure. The gas enclosed in a cylinder with a piston receives heat from the flame of a burner. On absorbing heat energy the gas expands and pushes the piston upwards. So the wheel starts rotating. To continue the rotations of the wheel an arrangement is done in the heat engine so that the piston can move up and down periodically. For this, when piston moves more in upward direction, then hot gas is released from the hole provided on upper side

Here gas is called working substance. The flame of the burner is called heat source and the arrangement in which gas is released is called heat sink.

Following figure shows working of the heat engines by line diagram

In the heat engine, the working substance undergoes a cyclic process. For this the working substance absorbs heat Q_1, from the heat source at higher temperature T, out of which a

part of energy is converted to mechanical energy (work W) and remaining heat Q_2 is released into the heat sink.
Hence , the net amount of heat absorbed by the working substance is
$Q = Q_1 - Q_2$
But for a cyclic process, the net heat absorbed by the system is equal to the net work done
$\therefore Q = W$
$Q_1 - Q_2 = W$
In the cyclic process, the ratio of the network (W) obtained during one cycle is called the efficiency (η) of the heat engine. That is

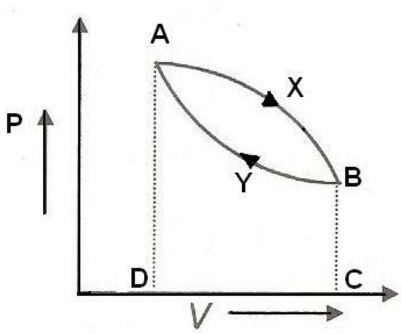
$$\eta = \frac{Net\ work\ obtaine\ per\ cycle}{Heat\ absorbed\ per\ cycle}$$

$$\eta = \frac{W}{Q_1} = \frac{Q_1 - Q_2}{Q_1}$$

$$\eta = 1 - \frac{Q_2}{Q_1} \quad ---eq(1)$$

From equation(1) it can be said that if $Q_2 = 0$, then the efficiency of the heat engine is $\eta = 1$. This means that the efficiency of heat engine becomes 100% and total heat supplied to the working substance gets completely converted into work.
In practice, for any engine $Q_2 \neq 0$ means that some heat Q_2 is always wasted hence $\eta<1$

Cyclic process and efficiency calculation



When a system after passing through various intermediate steps returns to its original state, then it is called a cyclic process.
Suppose a gas enclosed in cylinder is expanded from initial stage A to final stage B along path AXB as shown in figure
If W_1 be the work done by the system during expansion, then
W_1 = + Area AXBCDA
Now late the gas be compressed from state B to state along the path BYA, so as to return the system to the initial state. If W_2 be the work done on the system during compression, then W_2 = -Area BYADCB
According to sign convention, work done on the system during compression is negative and the net work done in the cyclic process AXBYA is
W = Area AXBCDA – Area BRADCB = Area AXBYA
Which is a positive quantity and hence net work will be done by the system
So the net amount of work done during a cyclic process is equal to the area enclosed by the cyclic path. It is evident from the figure that if the cyclic path is being traced in anticlockwise direction, the expansion curve will be below the compression curve and net

work done during the process will be negative. This implies that the net work will now be done on the system. Applying first law of thermodynamics to cyclic process, we get

$\Delta Q = \Delta U + \Delta W$

But ΔU = for cyclic process

So $\Delta Q = \Delta W$

Solved Numerical

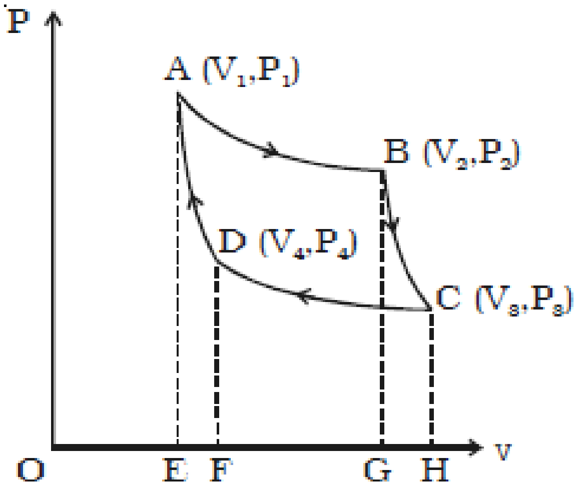
Q) An ideal monoatomic gas is taken round the cycle ABCD where co-ordinates of ABCDP_V diagram are A(p, V), B(2p, V), C(2p, 2V) and D(p, 2V). Calculate work done during the cycle

Solution Area enclosed = pV

Carnot Cycle and Carnot Engine

Carnot engine consists of a cylinder whose sides are perfect insulators of heat except the bottom and a piston sliding without friction. The working substance in the engine is μ mole of a gas at low enough pressure (behaving as an ideal gas). During each cycle of the engine, the working substance absorbs energy as heat from a heat source at constant temperature T_1 and releases energy as heat to a heat sink at a constant lower temperature $T_2 < T_1$.

The cyclic process, shown by P-V graph in figure a, is completed in four stages.



(a)

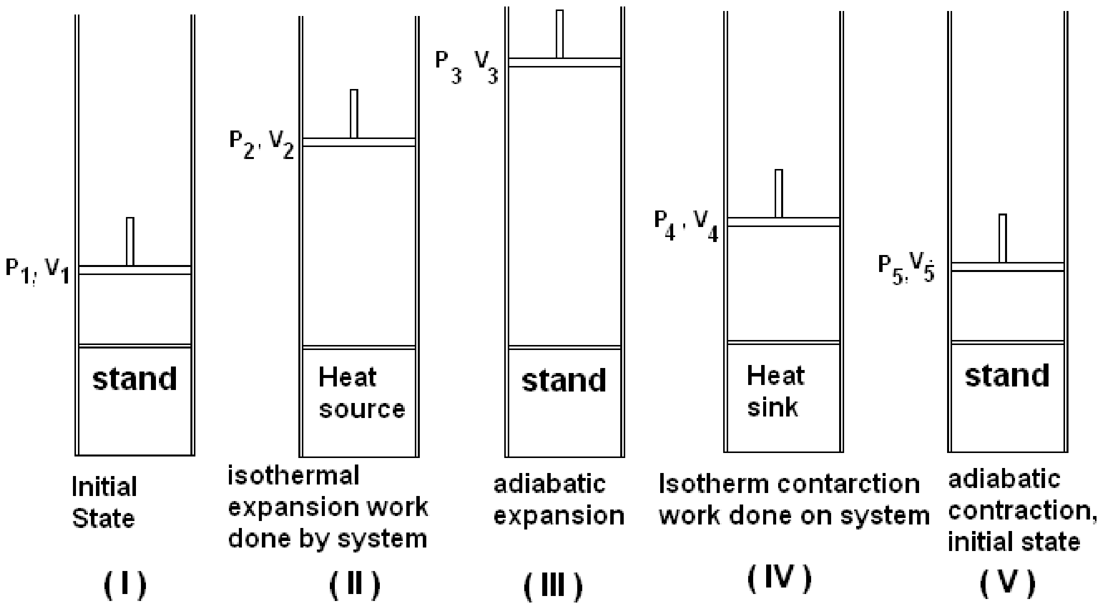
The Carnot engine and its different stages are shown in figure b

(I) First stage Isothermal expansion of gas from (a → b)

Initial equilibrium state (P_1, V_1, T_1)final equilibrium state (P_2, V_2, T_1) Suppose gas absorbs heat Q_1 during the process. Hence work done is

$$W_1 = Q_1 = \mu R T_1 \ln\left(\frac{V_2}{V_1}\right) \quad --- eq(1)$$

Further $P_1V_1 = P_2V_2$ -----------eq(2)



(b)

(II) Second stage Adiabatic expansion of gas (b → c)

Now , the cylinder is placed on a thermally insulated stand and the gas is adiabatically expanded to attain the state c (P_3, V_3, T_2).

During this (adiabatic process the gas does not absorb any heat but does work while expanding, so its temperature decreases. For this process

$$P_2V_2^{\gamma} = P_3V_3^{\gamma} \quad ---eq(3)$$

(III) Third Stage: Isothermal compression of gas (c→d)

Now, the cylinder is brought in contact with heat sink at temperature T_2 and isothermally compressed slowly to attain an equilibrium state d (P_4, V_4, T_2). Work done on the gas during this process of isothermal compression is negative as work is done on the gas from state c → d is

$$W_2 = Q_2 = -\mu RT_2 \ln\left(\frac{V_4}{V_1}\right)$$

$$W_2 = Q_2 = \mu RT_2 \ln\left(\frac{V_3}{V_4}\right) \quad ---eq(4)$$

Here Q_2 is released by the gas into heat sink

Further for isothermal process

$$P_3V_3 = P_4V_4 \text{-----------} eq(5)$$

(IV) Fourth Stage: Adiabatic compression of gas (d →a)

Now, the cylinder is placed on a thermally insulated stand and compressed adiabatically to its original state a ($P_1V_1T_1$). This process is adiabatic, therefore, there're is exchange of heat with surrounding, but the work is done on the gas and hence temperature increases from T_2 to T_1

For this adiabatic process

$$P_4V_4^{\gamma} = P_1V_1^{\gamma} \quad --- eq(6)$$

Note that over the whole cycle, the heat absorbed by the gas is Q_1 and the heat given out by the gas is Q_2. Hence the efficiency η of the Carnot engine is

$$\eta = 1 - \frac{Q_2}{Q_1}$$

From equation (1) and (4)

$$\eta = 1 - \frac{T_2 \ln\left(\frac{V_3}{V_4}\right)}{T_1 \ln\left(\frac{V_1}{V_4}\right)} \quad --- eq(7)$$

Multiplying equation (2), (3), (5) and 6 we get

$$P_1V_1P_2V_2^{\gamma}P_3V_3P_4V_4^{\gamma} = P_2V_2P_3V_3^{\gamma}P_4V_4P_1V_1^{\gamma}$$

$$\therefore (V_2V_4)^{\gamma-1} = (V_3V_1)^{\gamma-1}$$

$$\therefore V_2V_4 = V_3V_1$$

$$\frac{V_2}{V_1} = \frac{V_3}{V_4}$$

$$\therefore ln\left(\frac{V_2}{V_1}\right) = ln\left(\frac{V_3}{V_4}\right)$$

Using this result in equation (7) We get efficiency of Carnot engine as

$$\eta = 1 - \frac{T_2}{T_1} \quad --- eq(8)$$

Or

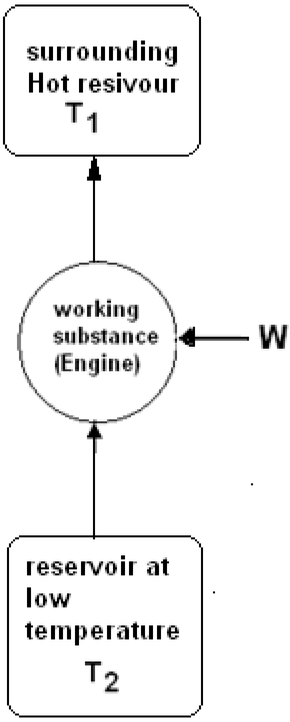
$$\eta = 1 - \frac{low\ temp\ of\ sink}{high\ temperature\ of\ source}$$

Equation (8) shows that the efficiency of the Carnot engine depends only on the temperature of the source and the sink. Its efficiency does not depends on the working substance (if it is ideal gas).

If the temperature of the source (T_1) is infinite or the temperature of the sink (T_2) is absolute zero 9 which is not possible) then only, the efficiency of Carnot engine will be 100%, which is impossible.

Refrigerator / Heat pump and Coefficient of Performance

If the cyclic process performed on the working substance in heat engine is reversed, then the system woks as a refrigerator or heat pump. Figure below shows the block diagram of refrigerator/ heat pump



In the refrigerator, the working substance absorbs heat Q_2 from the cold reservoir at lower temperature T_2, external work W, is performed on the working substance and the working substance releases heat Q_1 into the hot reservoir at higher temperature T_1

The ratio of the heat Q_1 absorbed by the working substance to the work W performed on it, is called the coefficient of performance (α) of the refrigerator. That is

$$\alpha = \frac{Q_2}{W}$$

Here heat is released in surrounding

$Q_1 = W + Q_2$

$Q_1 = W + Q_2$

$W = Q_1 - Q_2$

$$\alpha = \frac{Q_2}{Q_1 - Q_2}$$

Here the value of α can be more than 1 ($\because Q_2 > Q_1 - Q_2$), but it can not be infinite

TRANSFER OF HEAT

Transfer of heat

There are three ways in which heat energy may get transferred from one place to another. These are conduction, convection and radiation.

Conduction

Heat is transmitted through the solids by the process of conduction only. When one end of the solid is heated, the atoms or molecules of the solid at the hotter end becomes more strongly agitated and start vibrating with greater amplitude. The disturbance is transferred to the neighboring molecules.

Coefficient of thermal conductivity

Let us consider a metallic bar of uniform cross section A whose one end is heated. After sometime each section of the bar attains constant temperature but it is different at different sections. This is called steady state.

In this state there is no further absorption of heat.

If Δx is the distance between the two sections with a difference in temperature of ΔT and ΔQ is the amount of heat conducted in a time Δt, then it is found that the rate of conduction of heat is

(i) directly proportional to the area of cross section (A)

(ii) directly proportional to the temperature difference between the two sections (ΔT)

(iii) inversely proportional to the distance between the two sections (Δx).

$$\frac{\Delta Q}{\Delta T} \propto -A\frac{\Delta T}{\Delta x}$$

Negative sign indicates as x increases temperature decreases

$$\frac{\Delta Q}{\Delta T} = KA\frac{\Delta T}{\Delta x} \quad ----eq(1)$$

where K is a constant of proportionality called co-efficient of thermal conductivity of the metal

$\frac{\Delta T}{\Delta x}$ is called as temperature gradient

Coefficient of thermal conductivity of the material of a solid is equal to the rate of flow of heat per unit area per unit temperature gradient across the solid. Its unit is W m^{-1} K^{-1}.

Or Cal $s^{-1}m^{-1}K^{-1}$

Thermal steady state

When a rod is heated , after sufficiently long time the temperature of all parts of rod become steady. These steady temperatures decreases along the length of the rod from hot end to its cold end. In this situation amount of heat energy received by the hot end in some interval is equal to the amount of heat lost by the cold end in the same time interval. Hence, any cross section of the rod, along its entire length, has the same value of heat current dQ/dt. Further along the entire length of the rod the value of the temperature gradient dT/dx is also the same along the length.

Now both dQ/dt and dT/dx remains constant with time. This condition of the rod is called 'thermal steady state" of the rod.
In thermal steady state the temperature of two ends of the rod T_1 and T_2 with $T_1>T_2$. As dT/dx is same all along the length of the rod

$$\frac{dT}{dx} = -\left[\frac{T_1 - T_2}{L}\right]$$

From eq(1)

$$\frac{dQ}{dt} = KA\left[\frac{T_1 - T_2}{L}\right] \quad ----eq(2)$$

As dQ/dt is constant

$$\frac{Q}{t} = KA\left[\frac{T_1 - T_2}{L}\right]$$

$$Q = KA\left[\frac{T_1 - T_2}{L}\right] t$$

Above equation gives the amount of heat flowing through the rod in a steady thermal state in time t.

If we represent dQ/dt as heat current (H) , which caused due to temperature difference then from equation (2)

$$H = \left[\frac{T_1 - T_2}{\frac{L}{KA}}\right]$$

Comparing the above equation with I = V/R we get thermal resistance R_H

$$R_H = \frac{L}{KA}$$

Unit of thermal resistance is Kelvin/watt and its dimensional formula is $M^{-1}L^{-2}T^3K$

Formula for effective thermal resistance when thermal conductors are connected in series

$$(R_H)_S = (R_H)_1 + (R_H)_2$$

For parallel connection

$$\frac{1}{(R_H)_P} = \frac{1}{(R_H)_1} + \frac{1}{(R_H)_2}$$

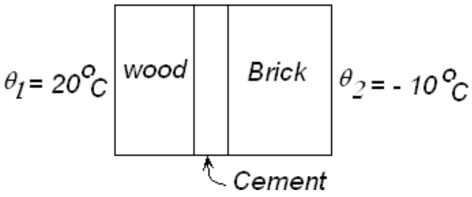
Solved Numerical

Q) An electric heater is sued in a room of total wall area 137 m^2 to maintain a temperature of 20^OC inside it, when the outside temperature is -10^OC. the walls have three different layers of materials. The innermost layer is of wood of thickness 2.5cm, the middle layer is of thickness 1.0cm and the outermost layer is of brick of thickness 25.0cm. Find the power of the electric heater. Assume that there is no heat loss through the floor and the ceiling.

The thermal conductivity of wood , cement and brick are 0.125 W/m ^{O}C, 1.5 W/m ^{O}C and 1.0 W/m ^{O}C respectively

Solution

Situation is as show in figure



The thermal resistance of wood, the cement and the brick layers are

$$R_W = \frac{L}{KA}$$

$$R_W = \frac{2.5 \times 10^{-2}}{0.125 \times 137} = \frac{0.2}{137}$$

$$R_C = \frac{1.0 \times 10^{-2}}{1.5 \times 137} = \frac{0.0067}{137}$$

$$R_B = \frac{25.0 \times 10^{-2}}{1.0 \times 137} = \frac{0.25}{137}$$

As the layers are connected in series, the equivalent

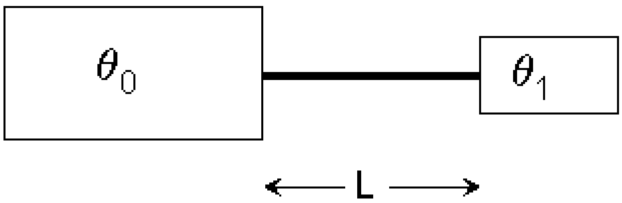
$$R = R_W + R_C + R_B$$

$$R = \frac{0.2 + 0.0067 + 0.25}{137} = 3.33 \times 10^{-3}$$

The heat current

$$i = \frac{\theta_1 - \theta_2}{R} = \frac{20 - (-10)}{3.33 \times 10^{-3}} = 9000\ W$$

Q) The figure shows a large tank of water at a constant temperature θ_0 and a small vessel containing a mass m of water at an initial temperature θ_1 ($<\theta_0$). A metal rod of length L, area of cross-section A and thermal conductivity K connects the two vessels. Find the time taken for the temperature of water in the smaller vessel to become θ_2 ($\theta_1 < \theta_2 < \theta_0$). Specific heat capacity of water is s and all other heat capacities are negligible.



Solution

Suppose, the temperature of the water in the smaller vessel is θ at time t. In the next time interval dt, a heat ΔQ is transferred from the big vessel

$$\Delta Q = \frac{KL}{A}(\theta_0 - \theta)dt \quad ----eq(1)$$

This heat increases the temperature of the water in small tank to θ+dθ where

$$\Delta Q = ms\, d\theta \text{ ----------- } eq(2)$$

From equation (1) and (2)

$$ms\, d\theta = \frac{KL}{A}(\theta_0 - \theta)dt$$

$$dt = \frac{Lms}{KA}\frac{d\theta}{\theta_0 - \theta}$$

Or

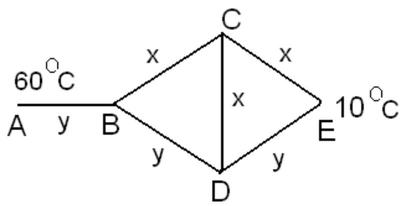
$$\int_{0}^{T} dt = \frac{Lms}{KA} \int_{\theta_1}^{\theta_2} \frac{d\theta}{\theta_0 - \theta}$$

Where T is the time required for the temperature of the water to become θ_2

Thus

$$T = \frac{Lms}{KA} ln \left(\frac{d\theta}{\theta_0 - \theta}\right)$$

Q) three rods of material x and three rods of material y are connected as shown. All the rods are of identical length and cross sectional area. If the end A is maintained at 60^{O}c and the junction E is at 10^{O}C, calculate the temperature of the junctions B,C and D. The thermal conductivity of x is 0.92c.g.s unit and that of y is 0.46c.g.s units



Solution:

Since end A or rod AB is maintained at temperature higher than the end b heat is conducted from A to B

Now the total heat entering junction B is equal to the total heat leaving it (all by conduction alone)

Let the temperature of junction B, C, D be T_1, T_2 and T_3 respectively

Let the cross-sectional area of each rod be A and the length of rod be L. Then heat entering joint B per second =

$$= \frac{K_x A(60 - T_1)}{L}$$

Heat leaving B per second = heat passing through BC + heat passing through BD

$$\frac{K_x A(T_1 - T_2)}{L} + \frac{K_y A(T_1 - T_3)}{L}$$

Thus

$$\frac{K_x A(60 - T_1)}{L} = \frac{K_x A(T_1 - T_2)}{L} + \frac{K_y A(T_1 - T_3)}{L}$$

Given $K_X = 2K_Y$

$60 - T_1 = 2 (T_1 - T_2) + (T_1 - T_2)$

Or $4T_1 - 2T_2 - T_3 = 60$ ------------ eq(1)

Similarly for junction c

Heat received per second = Heat passing through CD + Heat passing through CE

$$\frac{K_x A(T_1 - T_2)}{L} = \frac{K_x A(T_2 - 10)}{L} + \frac{K_x A(T_2 - T_3)}{L}$$

OR $T_1 - T_2 = T_2 - 10 + T_2 - T_3$

Or $T_1 - 3T_2 + T_3 = -10$ ------------- eq(2)

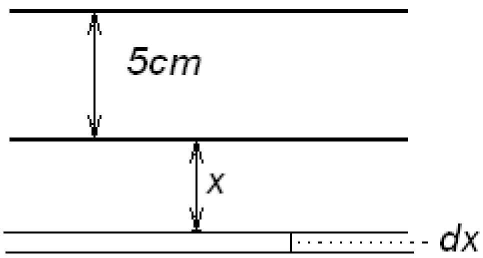
For Junction D

$$\frac{K_y A(T_1 - T_2)}{L} = \frac{K_x A(T_3 - T_2)}{L} + \frac{K_y A(T_3 - 10)}{L}$$

$T_1 - T_2 = 2(T3 - T_2) + T_3 - 10$
$T_1 + 2T_2 - 4T_3 = -10$ ---------- eq(3) Solving equation (1) (2) and (3)
we get $T_1 = 30^OC$, $T_2 = 20^O$, $T_3 = 20^OC$
Temperature of junction B = 30^OC
Temperature of junction C = 20^OC
Temperature of junction D = 20^OC

Q) The thickness of ice layer on the surface of lake is 5cm. Temperature of environment is -10^OC. Find the time required for the thickness of the ice layer to become double. Thermal conductivity of ice is 0.004 cal/cms ^OC, density of ice is 0.92 g/cm^3 and latent heat of fusion is 80cal/gm

Solution



Consider a layer of thickness dx and surface area A.
Heat required to b taken out from such layer is
dQ = Adx ρ L
Here ρ is density of ice and L is latent heat of melting
If time required for passage of heat through a thickness of 5x is dt Then

$$dQ = KA\frac{\Delta T}{5+x}dt$$

Thus

$$KA\frac{\Delta T}{5+x}dt = \mathrm{Adx}\ \rho\ \mathrm{L}$$
$$K(-10-0)dt = \mathrm{A(5+x)dx}\ \rho\ \mathrm{L}$$

Integrating

$$-10K\int_0^t dt = \rho\ \mathrm{L}\int_0^5 (5+\mathrm{x})\mathrm{dx}$$

$$-10Kt \ = \ \rho L\ \{[5x]_0^5 + [\frac{x^2}{2}]_0^5\ \}$$

$$-10Kt \ = \ \rho L(25+12.5)$$
$$-10 \times 0.004 \times t = 0.92 \times 80 \times (37.5)$$

t= 69,000 seconds

t = 19.16 hours

Thermal expansion

The increase in dimension of a substance due to absorption of heat is called thermal expansion and decrease in dimensions of the substance by releasing the heat is called thermal contraction

Linear expansion

The increase in the length of a body with increase in temperature is called linear expansion

For small change in temperature, the increase in length $\Delta\ell$ is directly proportional to original length ℓ and increase in temperature ΔT

$\Delta\ell \propto \ell$ and $\Delta\ell \propto \Delta T$

$\Delta\ell = \alpha\,\ell$

Here α is a constant of proportionality called coefficient of linear expansion of material of the body. The value of α depends on the type of material of body and temperature. If temperature interval is very large , then α do not depends on the temperature

The unit of α is $(^{O}C)^{-1}$ or K^{-1}

$\ell' = \ell\,(1+ \alpha\,\Delta T)$

Some substances exhibits uniform thermal expansion in all directions. Such substances are called isotropic substances. For such substance

Increase in area $\Delta A = 2\alpha A\Delta T$

Increase in volume $\Delta V= \gamma\Delta T$

$V' = V(1+ \gamma\alpha)$

For density

$P'=\rho(1+ \gamma\alpha)$

Thermal expansion is more in liquid than solid and it is maximum in gases

Solved Numerical

Q) The design of some physical instrument requires that there be a constant difference in length of 10cm between ion rod and copper rod laid side at all temperatures. Find their length ($\alpha_{Fe} = 11\times10^{-6}\ {}^{O}C^{-1}$, $\alpha_{Cu} = 17\times10^{-6}\ {}^{O}C^{-1}$)

Solution:

Since $\alpha_{Cu} > \alpha_{Fe}$ so length of iron rod should be greater than the length of cooper rod

Let initial length of iron rod be l_1 and copper rod be l_2 then

$l_1 - l_2 = 10cm$ -------------- eq(1)

Also since the difference has to be constant at all the temperatures, so

$\Delta l = l_1\,\alpha_{Fe}\Delta T = l_2\,\alpha_{Cu}\,\Delta T$

$$\frac{l_1}{l_2} = \frac{\alpha_{Cu}}{\alpha_{Fe}} \qquad ---eq(2)$$

Solving equation (1) and (2), we get

$l_1 = 28.3$ cm and $l_2 = 18.3$ cm

Q) A sphere of diameter 7.0cm and mass 266.5 g floats in a bath of liquid . As a temperature is raised, the sphere begins to sink at a temperature of $35^{O}C$. If the density of the liquid is 1.527 g/cm^2 at $0^{O}C$, find the coefficient of cubical expansion of the liquid. Neglect the expansion of the sphere

Solution

It is given that the expansion of the sphere is negligible as compared to the expansion of liquid. At $0^{O}C$, the density of the liquid is $\rho_0 = 1.527$ g/cm^3. At $35^{O}C$, the density of the liquid equals the density of the sphere. Thus

$$\rho_{35} = \frac{266.5}{\frac{4}{3}\pi(3.5)^2} = 1.484\ g/cm^3$$

We have density ρ ∝ (1/V) thus

$$\frac{\rho_\theta}{\rho_0} = \frac{V_0}{V_\theta} = \frac{1}{(1+\gamma\theta)}$$

Or

$$\rho_\theta = \frac{\rho_0}{(1+\gamma\theta)}$$

$$\gamma = \frac{\rho_0 - \rho_{35}}{\rho_{35}(35)} = \frac{(1.527 - 1.484)}{1.484 \times 35}$$

$$\gamma = 8.28\times10^{-4}\ /\ ^{O}C$$

Convection

It is a phenomenon of transfer of heat in a fluid with the actual movement of the particles of the fluid. When a fluid is heated, the hot part expands and becomes less dense. It rises and upper colder part replaces it. This again gets heated, rises up replaced by the colder part of the fluid. This process goes on.

This mode of heat transfer is different from conduction where energy transfer takes place without the actual movement of the molecules.

Application

It plays an important role in ventilation and in heating and cooling system of the houses.

Radiation

It is the phenomenon of transfer of heat without any material medium. Such a process of heat transfer in which no material medium takes part is known as radiation.

Thermal radiation

The energy emitted by a body in the form of radiation on account of its temperature is called thermal radiation. It depends on,

(i) temperature of the body,

(ii) nature of the radiating body

The wavelength of thermal radiation ranges from 8 × 10-7 *m* to 4 × 10-4 *m*. They belong to infra-red region of the electromagnetic spectrum.

Properties of thermal radiations

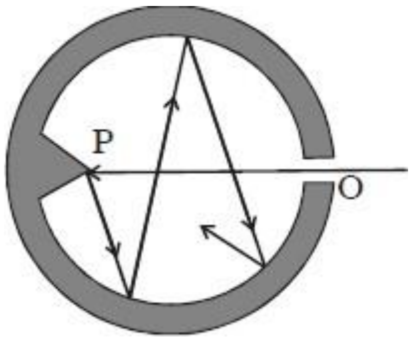
1. Thermal radiations can travel through vacuum.
2. They travel along straight lines with the speed of light.
3. They can be reflected and refracted. They exhibit the phenomenon of interference and diffraction.
4. They do not heat the intervening medium through which they pass.
5. They obey inverse square law.

Absorptive and Emissive power

Absorptive power
Absorptive power of a body for a given wavelength and temperature is defined as the ratio of the radiant energy absorbed per unit area per unit time to the total energy incident on it per unit area per unit time. It is denoted by a_λ.
Emissive power
Emissive power of a body at a given temperature is the amount of energy emitted per unit time per unit area of the surface for a given wavelength. It is denoted by e_λ.
Its unit is W m^{-2}.

Perfect black body
A perfect black body is the one which absorbs completely heat radiations of all wavelengths which fall on it and emits heat radiations of all wavelengths when heated. Since a perfect black body neither reflects nor transmits any radiation, the absorptive power of a perfectly black body is unity.

Fery's black body



Fery's black body consists of a double walled hollow sphere having a small opening O on one side and a conical projection P just opposite to it ref.figure Its inner surface is coated with lamp black. Any radiation entering the body through the opening O suffers multiple reflections at its inner wall and about 97% of it is absorbed by lamp black at each reflection. Therefore, after a few reflections almost entire radiation is absorbed. The
projection helps in avoiding any direct reflections which even otherwise is not possible because of the small opening O. When this body is placed in a bath at fixed temperature, the heat radiations come out of the hole. The opening O thus acts as a black body radiator.

Kirchoff's Law
According to this law, the ratio of emissive power to the absorptive power corresponding to a particular wavelength and at a given temperature is always a constant for all bodies. This constant is equal to the emissive power of a perfectly black body at the same temperature and the same wavelength. Thus, if e_λ is the emissive power of a body corresponding to a wavelength λ at any given temperature, a_λ is the absorptive power of the body corresponding to the same wavelength at the same temperature and E_λ is the emissive power of a perfectly black body corresponding to the same wavelength and the same temperature, then according to Kirchoff's law

$$\frac{a_\lambda}{e_\lambda} = constant = E_\lambda$$

From the above equation it is evident that if a_λ is large, then e_λ will also be large (i.e) if a body absorbs radiation of certain wavelength strongly then it will also strongly emit the radiation of same wavelength. In other words, good absorbers of heat are good emitters also.

Applications of Kirchoff's law

(i) The silvered surface of a thermos flask is a bad absorber as well as a bad radiator. Hence, ice inside the flask does not melt quickly and hot liquids inside the flask do not cool quickly.

(ii) Sodium vapours on heating, emit two bright yellow lines. These are called D_1 and D_2 lines of sodium. When continuous white light from carbon arc passes through sodium vapour at low temperature, the continuous spectrum is absorbed at two places corresponding to the wavelengths of D_1 and D_2 lines and appear as dark lines. This is in accordance with Kirchoff's law.

Wien's displacement law

Wien's displacement law states that the wavelength of the radiation corresponding to the maximum energy (λ_m) decreases as the temperature T of the body increases.

(i.e) $\lambda_m T = b$ where b is called Wien's constant. Its value is 2.898×10^{-3} m K

Stefan's law

Stefan's law states that the total amount of heat energy radiated per second per unit area of a perfect black body is directly proportional to the fourth power of its absolute temperature.

(i.e) $E \propto T^4$ or $E = \sigma e_\lambda T^4$

where σ is called the Stefan's constant. Its value is 5.67×10^{-8} W m^{-2} K^{-4}.

It is also called Stefan - Boltzmann law, as Boltzmann gave a theoretical proof of the result given by Stefan.

If the body of temperature T is kept in an environment with temperature T_s ($T > T_S$), then rate at which the body loses heat is given by

$$\frac{dQ}{dt} = e_\lambda \sigma A(T^4 - T_S^4)$$

A : is area of surface , $e_\lambda = 1$ for perfectly black body

Solved numerical

Q) From 1 m^2 area of surface of Sun 6.3×10^7 J energy is emitted per second $\sigma = 5.669\times10^{-8}$ $Wm^{-2}K^{-4}$. Find the temperature of the surface of the sun

Solution.

A = 1 m^2 , dQ/dt = 6.3×10^7 J/s, $e_\lambda = 1$

$$\frac{dQ}{dt} = e_\lambda \sigma A T^4$$

$6.3\times10^7 = 5.669\times10^{-8}\times1\times1\times T^4$

T = 5841 K

Q) How many times faster the temperature of a cup of tea will decrease by 1^OC at 373K, then at 303K? Consider tea as a black body. Take room temperature as 293K

($\sigma = 5.7\times10^{-8}$ W m^{-2} K^{-4})
Solution

$$\frac{dQ_1}{dt} = e_\lambda \sigma A(373^4 - 293^4)$$
$$\frac{dQ_2}{dt} = e_\lambda \sigma A(303^4 - 293^4)$$

Take the ratio of above equations we get cup at 373K will decrease its temperature 11.32 times faster than cup at 303K temperature

Newton's law of cooling

Newton's law of cooling states that *the rate of cooling of a body is directly proportional to the temperature difference between the body and the surroundings.*

The law holds good only for a small difference of temperature. Loss of heat by radiation depends on the nature of the surface and the area of the exposed surface.

We know that the amount of heat required to change the temperature of a bosy of mass m and specific heat c, by ΔT is

ΔQ = mcΔT

Therefore, the rate of loss of heat

$$\frac{dQ}{dt} = -mc\frac{dT}{dt}$$

According to Newton's law, the rate of loss of heat by a body depends on the difference of temperature ($T - T_S$) between body and its surrounding

$$\therefore \frac{dQ}{dt} = -mc\frac{dT}{dt} \propto (T - T_S)$$
$$\therefore \frac{dQ}{dt} = -k'(T - T_S)$$

Here dT/dt is the rate of decrease in temperature of a body at temperature T.

The constant k' depends on the mass and the specific heat of the cooling body. Negative sign indicates temperature of body decreases with time

Note that Newton's law of cooling is true only for small interval of diierence of temperature between the body and its surrounding

If the amount of heat lost by the body due to radiation is very small, this law hold true for large interval of temperature also.

For natural convection, the law of cooling given by Langmuir – Lorenz is as under

$$-\frac{dT}{dt} \propto (T - T_S)^{\frac{5}{4}}$$

Solved Numerical

Q) A body at 80^OC cools down to 64^OC in 5 minutes and in 10 minutes it cools down to 52^OC. What will be its temperature after 20 minutes? What is the temperature of the environment

Solution

For the first 5 minute

$\Delta T = T_2 - T_1 = 64 - 80 = -16$ and $\Delta t = 5$

$$\therefore \frac{+16}{5} = k' \left(\frac{80+64}{2} - T_s\right) \quad ----eq(1)$$

Here we have taken the average of initial and final temperatures as the temperature of the body

Similarly for next 5 minutes

$\Delta T = 52 - 64 = -12$

$$\therefore \frac{+12}{5} = k' \left(\frac{52+64}{2} - T_s\right) \quad ----eq(2)$$

Dividing eq(1) by eq(2) we get

$$\frac{16}{5} \times \frac{5}{12} = \frac{72 - T_S}{58 - T_S}$$

$232 - 4T_S = 216 - 3T_S$

$232 - 216 = T_S$

$T_S = 16^{O}C$

Substituting value of T_S in equation(1)

$$\frac{16}{5} = k'(72 - 16)$$

K' = 2/35

Now for third stage $\Delta t = 10$ minutes

$\Delta T = T - 52$, where T is final temperature

$$\frac{52 - T}{10} = \frac{2}{35}\left(\frac{52 + T}{2} - 16\right)$$

On solving we get

$T = 36^{O}C$

Wave Optics

Wave front

The wave front at any instant is defined as the locus of all the particles of the medium which are in the same state of vibration.

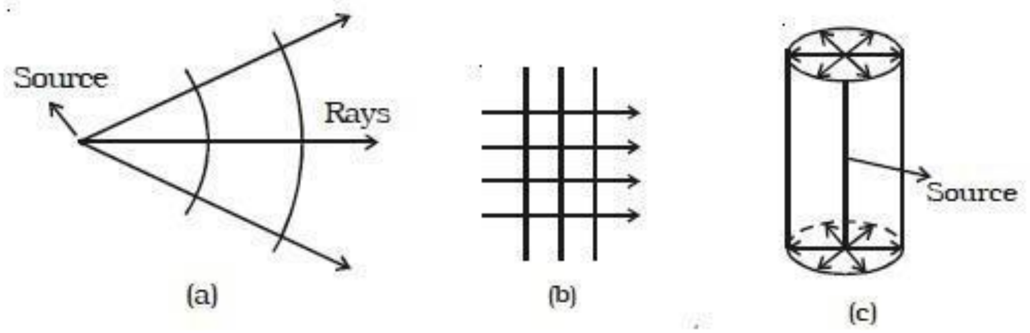
Or

Aan imaginary surface passing through particles oscillating with same phase is known as wavefront

A point source of light at a finite distance in an isotropic medium emits a spherical wave front (Fig a).

A point source of light in an isotropic medium at infinite distance will give rise to plane wavefront (Fig. b).

A linear source of light such as a slit illuminated by a lamp, will give rise to cylindrical wavefront (Fig c).



HUYGENS PRINCIPLE

Huygen's principle states that,

(i) every point on a given wave front may be considered as a source of secondary wavelets which spread out with the speed of light in that medium and

(ii) the new wavefront is the forward envelope of the secondary wavelets at that instant

Huygen's construction for a spherical and plane wavefront:

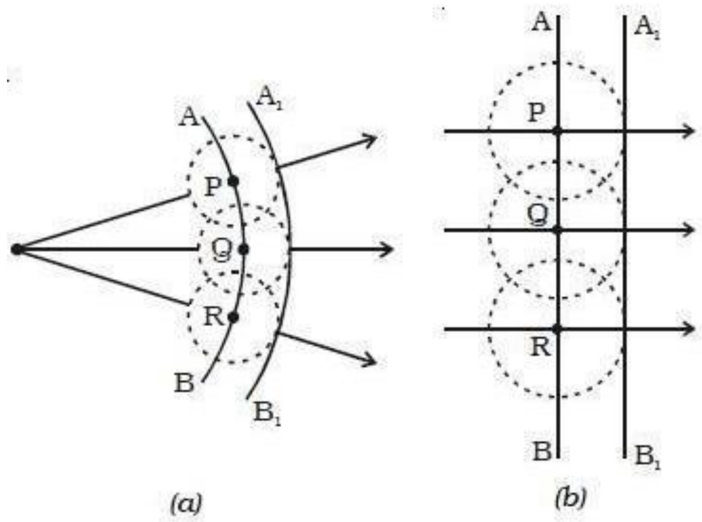
Huygen's construction for a spherical and plane wavefront is shown in Fig.a.

Let AB represent a given wavefront at a time t = 0. According to Huygen's principle, every point on AB acts as a source of secondary wavelets which travel with the speed of light c.

To find the position of the wave front after a time t, circles are drawn with points P, Q, R ... etc as centres on AB and radii equal to ct.

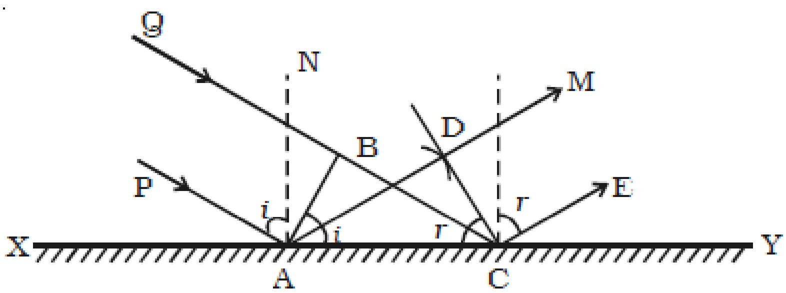
These are the traces of secondary wavelets. The arc A_1B_1 drawn as a forward envelope of the small circles is the new wavefront at that instant.

If the source of light is at a large distance, we obtain a plane wave front A_1 B_1 as shown in Fig b.



Reflection of a plane wave front at a plane surface

Let XY be a plane reflecting surface and AB be a plane wavefront incident on the surface at A. PA and QBC are perpendiculars drawn to
AB at A and B respectively. Hence they represent incident rays. AN is the normal drawn to the surface.
The wave front and the surface are perpendicular to the plane of the paper (Fig.).
According to Huygen's principle each point on the wavefront acts as the source of secondary wavelet.
By the time, the secondary wavelets from B travel a distance BC, the secondary wavelets from A on the reflecting surface would travel the same distance BC after reflection.
Taking A as centre and BC as radius an arc is drawn.
From C a tangent CD is drawn to this arc. This tangent CD not only envelopes the wavelets from C and A but also the wavelets from all the points between C and A.



Therefore CD is the reflected plane wavefront and AD is the reflected ray.

Laws of reflection

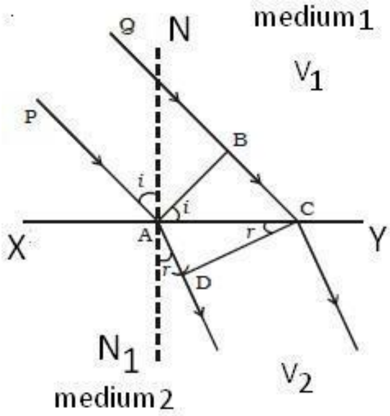
(i) The incident wavefront AB, the reflected wavefront CD and the reflecting surface XY all lie in the same plane.
(ii) Angle of incidence i = ∠ PAN = 90^0 − ∠ NAB = ∠BAC

Angle of reflection r = ∠ NAD = 90^0 − ∠ DAC = ∠DCA

$\angle B = \angle D = 90^{O}$
BC = AD and AC is common
∴ The two triangles are congruent
∠ BAC = ∠DCA
i.e. i = r
Thus the angle of incidence is equal to angle of reflection.

Refraction of a plane wavefront at a plane surface

Let XY be a plane refracting surface separating two media 1 and 2 of refractive indices μ_1 and μ_2 (Fig). The velocities of light in these two media are respectively v_1 and v_2.
Consider a plane wave front AB incident on the refracting surface at A. PA and QBC are perpendiculars drawn to AB at A and B respectively. Hence they represent incident rays. NAN_1 is the normal drawn to the surface. The wave front and the surface are perpendicular to the plane of the paper.
According to Huygen's principle each point on the wave front act as the source of secondary wavelet.
By the time, the secondary wavelets from B, reaches C, the secondary wavelets from the point A would travel a distance AD = v_2t, where t is the time taken by the wavelets to travel the distance BC.
∴BC = C_1t and AD = C_2t
Taking A as centre and C_2t as radius an arc is drawn in the second medium. From C a tangent CD is drawn to this arc.
.
Therefore CD is the refracted plane wavefront and AD is the refracted ray



Laws of refraction

(i) The incident wave front AB, the refracted wave front CD and the refracting surface XY all lie in the same plane.
(ii) From figure for Δ ABC and Δ ACD

$$\frac{\sin i}{\sin r} = \frac{BC/_{AC}}{AD/_{AC}} = \frac{BC}{AD} = \frac{v_1 t}{v_2 t} = \frac{v_1}{v_2} = n_{21}$$

Constant n_{21} in above equation is known as refractive index of medium 2 with respect to medium also represented as ${}_1\mu_2$

This is Snell's law of refraction

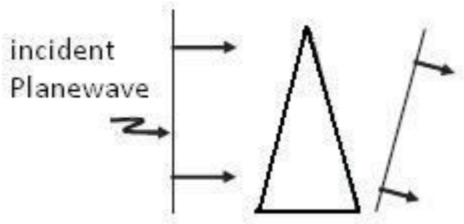
Further, if λ_1 and λ_2 denote the wavelengths of light in medium 1 and medium 2, respectively and if the distance BC is equal to λ_1 then the distance AE will be equal to λ_2 (because if the crest from B has reached C in time τ, then the crest from A should have also reached *E* in time τ); thus

$$\frac{\lambda_1}{\lambda_2} = \frac{BC}{AE} = \frac{v_1}{v_2}$$

The above equation implies that when a wave gets refracted into a denser medium ($v_1 > v_2$) the wavelength and the speed of propagation decrease but the *frequency* f (= v/λ) *remains the same*.

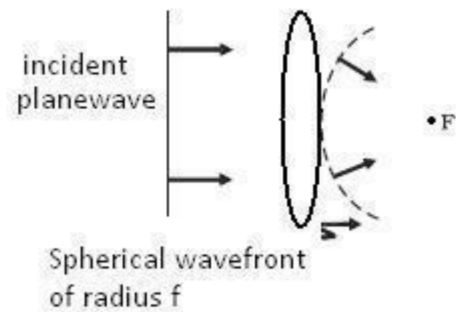
Refraction of a plane wave by a thin prism

we consider a plane wave passing through a thin prism. Clearly, since the speed of light waves is less in glass, the lower portion of the incoming wavefront (which travels through the greatest thickness of glass) will get delayed resulting in a tilt in the emerging wavefront as shown in the figure.



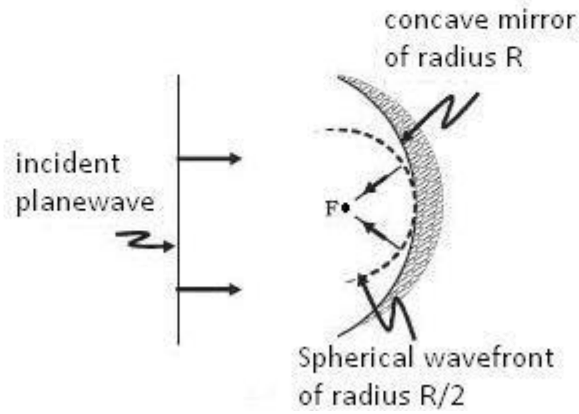
(b) a convex lens.

We consider a plane wave incident on a thin convex lens; the central part of the incident plane wave traverses the thickest portion of the lens and is delayed the most. The emerging wavefront has a depression at the centre and therefore the wavefront becomes spherical and converges to the point F which is known as the *focus*.



(c) Reflection of a plane wave by a concave mirror

a plane wave is incident on a concave mirror and on reflection we have a spherical wave converging to the focal point F.

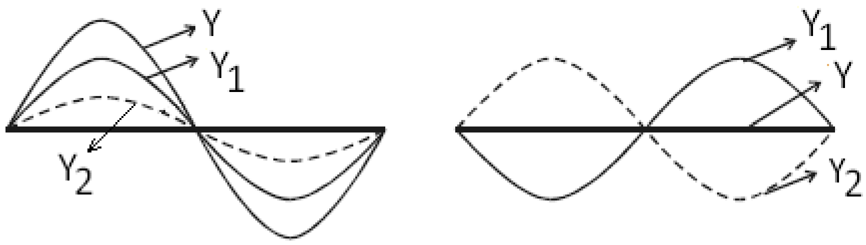


Coherent and incoherent sources

Two sources are said to be coherent if they emit light waves of the same wave length and start with same phase or have a constant phase difference.
Two independent monochromatic sources, emit waves of same wave length. But the waves are not in phase. So they are not coherent.
This is because, atoms cannot emit light waves in same phase and these sources are said to be incoherent sources.

Superposition principle

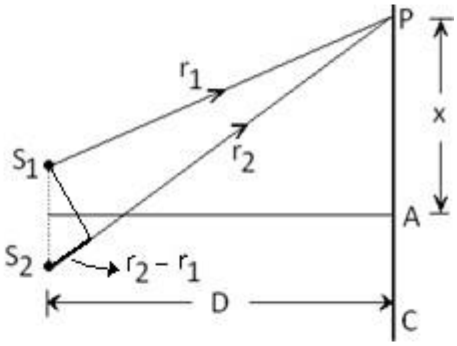
When two or more waves simultaneously pass through the same medium, each wave acts on every particle of the medium, as if the other waves are not present. The resultant displacement of any particle is the vector addition of the displacements due to the individual waves. This is known as principle of superposition. If $\mathbf{Y_1}$ and $\mathbf{Y_2}$ represent the individual displacement then the resultant displacement is given by $\mathbf{Y} = \mathbf{Y_1} + \mathbf{Y_2}$



Thus, superposition principle describes a situation when more than one waves superpose (i.e. interfere) at a point.
“ The effect produced by superposition of two or more wave is called interference”.

Interference due to two waves

Suppose two harmonic waves having initial phase φ_1 and φ_2 are emitted from two point like sources S_1 and S_2 respectively. They superimpose simultaneously (i.e. at the same time t) at a point P as shown in figure.



Visible perception of light is produced only by electric field, and therefore, in the present case we write light waves produced by source S_1 and S_2 in terms of electric fields (**e**) only.
Due to S_1 source,

$$\vec{e}_1 = \vec{E}_1 sin(\omega_1 t - k_1 r_1 + \varphi_1)$$

And due to source S_2

$$\vec{e}_2 = \vec{E}_2 sin(\omega_2 t - k_2 r_2 + \varphi_2)$$

Here $\mathbf{E_1}$ and $\mathbf{E_2}$ represent amplitude of electric fields, ω_1 and ω_2 denotes angular frequencies of waves, and k_1 and k_2 are wave vectors.
Let $\delta_1 = \omega_1 t - k_1 r_1 + \varphi_1$ and $\delta_2 = \omega_2 t - k_2 r_2 + \varphi_2$
Then $\mathbf{e_1} = \mathbf{E_1} \sin \delta_1$ and $\mathbf{e_2} = \mathbf{E_2} \sin \delta_2$
Now according to principle of superposition
$\mathbf{e} = \mathbf{e_1} + \mathbf{e_2}$
magnitude of resultant vector **e** $\quad e^2 = e_1^2 + e_2^2 + 2\vec{e}_1 \vec{e}_2$

If at a instant of time E_1 and E_2 amplitude of waves then, resultant amplitude E is

$$E^2 = E_1^2 + E_2^2 + 2E_1E_2 cos(\delta_1 - \delta_2)$$

The average intensity of light is proportional to square of amplitude
$I \propto E^2$ thus equation becomes

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\langle cos(\delta_1 - \delta_2)\rangle$$

In above equation I_1 and I_2 are the average intensities due to each wave. They are independent of time.
The last term in above equation is known as *interference term* which depends on time
Now

$$\langle cos(\delta_1 - \delta_2)\rangle = \frac{1}{T}\int_{t=0}^{t=T} cos\,(\delta_1 - \delta_2)dt$$

Here t is period of electric field oscillation. On substituting value of δ_2 and δ_1 in above equation

$$\langle cos(\delta_1 - \delta_2)\rangle = \frac{1}{T}\int_{t=0}^{t=T} cos\,\{(\omega_1 t - \omega_2 t) + (k_2 r_2 - k_1 r_1) + (\varphi_2 - \varphi_1)\}dt \text{ --(1)}$$

Case I: Incoherent sources
If angular frequency of both source is not same thus $\cos(\delta_1 - \delta_2)$ is time dependent and average value is zero. Thus superposed two waves produce the average intensity $I_1 + I_2$ at point P
Case II: Coherent sources:
For sources to be coherent there angular frequency should be same thus $\omega_1 = \omega_2 = \omega$ (say) Also since both waves are travelling in same medium there speed will be also same thus wave length is same thus $k_1 = k_2 = k$ (say) for sake of simplicity we will consider $\varphi_2 = \varphi_1$.
From equation (1) ignoring negative sign of cos

$$\langle cos(\delta_1 - \delta_2)\rangle = \frac{1}{T}\int_0^T cos\{k(r_2 - r_1)\}dt$$

$$\langle cos(\delta_1 - \delta_2)\rangle = \frac{1}{T}cos\{k(r_2 - r_1)\}\int_0^T dt$$

$\langle cos(\delta_1 - \delta_2)\rangle = cos\{k(r_2 - r_1)\}$-- eq(2)
Further we will assume that amplitude of both waves is equal $I_1 = I_2 = I'$ then
From equation (1) and eq(2) we get

$$I = I' + I' + 2\sqrt{I'I'}cosk(r_2 - r_1)$$

$$I = 2I' + 2I'cosk(r_2 - r_1)$$

$$I = 2I'\{1 + cosk(r_2 - r_1)\}$$

$$I = 2I'[2cos^2\{\frac{k(r_2 - r_1)}{2}\}]$$

[Use trigonometric identity $cos^2u = \frac{1+cos2u}{2}$]

$$I = 4I'cos^2\{\frac{k(r_2 - r_1)}{2}\}$$

Here $r_2 - r_1 = \delta$ is known as the path difference between superposing waves

$$I = 4I'cos^2\{\frac{k\delta}{2}\}$$

Special Cases

Case I : Constructive Interference
For $I = 4I' = I_0$ maximum intensity of light

term $cos^2\{\frac{k(r_2-r_1)}{2}\}$ Should be equal to one, It is possible if

$$\frac{k\delta}{2} = n\pi$$

$$\frac{2\pi\delta}{2\lambda} = n\pi \quad \because k = \frac{2\pi}{\lambda}$$

$$\delta = n\lambda$$

Here n = 0, 1, 2, 3, ……

"If the path difference between superposing waves is nλ (n = 0, 1, 2, 3, …..) intensity at a superposing point is maximum. Such interference is called constructive interference"

From equation

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\langle cos(\delta_1 - \delta_2)\rangle$$

For constructive interference $\cos(\delta_2-\delta_1) = 0$ thus

$$I_{max} = I_1 + I_2 + 2\sqrt{I_1 I_2}$$

Or

$$I_{max} = (\sqrt{I_1} + \sqrt{I_2})^2$$

$$I_{max} \propto (A_1 + A_2)^2$$

Case II: Destructive Interference

For intensity I = 0

term $cos^2\{\frac{k(\delta)}{2}\}$ Should be equal to zero, It is possible if

$$\frac{k\delta}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \ldots ....$$

Or

$$\frac{k\delta}{2} = (2n-1)\frac{\pi}{2}$$

As $k = 2\pi/\lambda$

$$\frac{2\pi\delta}{2\lambda} = (2n-1)\frac{\pi}{2}$$

$$\delta = (2n-1)\frac{\lambda}{2}$$

Here n = 1, 2, 3, 4, …..

" If phase difference between superposing waves is (2n-1)π intensity at a superposing point is minimum. This interference is called destructive interference"

"If path difference between superposing wave is (2n-1) (λ/2) intensity at superposed point is minimum. Such interference is known as destructive interference"

Here n = 1, 2, 3, 4, ….

From equation

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\langle cos(\delta_1 - \delta_2)\rangle$$

For destructive interference cos($\delta_1 - \delta_2$) = -1 thus

$$I_{min} = I_1 + I_2 - 2\sqrt{I_1 I_2}$$

Or

$$I_{min} = (\sqrt{I_1} - \sqrt{I_2})^2$$

$$I_{min} \propto (A_1 - A_2)^2$$

Condition for sustained interference

The interference pattern in which the positions of maximum and minimum intensity of light remain fixed with time, is called sustained or permanent interference pattern. The conditions for the formation of sustained interference may be stated as :

(i) The two sources should be coherent

(ii) Two sources should be very narrow

(iii) The sources should lie very close to each other to form distinct and broad fringes

Solved numerical

Q) Two sources of intensity I and 3I are used in an interference experiment. Find the intensity at a point where the waves from the two sources superimpose with a phase difference (1) Zero (2) π/2

Solution:

In case of interference

$$I' = I_1 + I_2 + 2\sqrt{I_1 I_2}\langle cos(\delta_1 - \delta_2)\rangle$$

$$I' = I_1 + I_2 + 2\sqrt{I_1 I_2}\langle cos(\delta)\rangle$$

(1) As δ = 0, cosδ =1

$$\therefore I' = 3I + I + 2\sqrt{(3I)(I) \times 1}$$

$$I' = 4I + 2\sqrt{3}I$$

(2) As δ = π/2, cosδ = 0

$$\therefore I' = 3I + I + 2\sqrt{(3I)(I) \times 0}$$

$$\therefore I' = 4I$$

Q) Ratio of the intensities of rays emitted from two different coherent sources is α. For the interference pattern formed by them, prove that

$$\frac{I_{max} + I_{min}}{I_{max} - I_{min}} = \frac{1 + \alpha}{2\sqrt{\alpha}}$$

I_{max} : Maximum of intensity in the interference fringe

I_{min} : Minimum of intensity in the interference fringe

Solution:

Given $I_1 = \alpha I_2$
Since $I \propto A^2$
Thus $A_1 = \sqrt{\alpha}\, A_2$
Now

$$I_{max} \propto (A_1 + A_2)^2$$

$$I_{max} \propto (\sqrt{\alpha}A_2 + A_2)^2$$

$$I_{max} \propto A_2^2(\sqrt{\alpha} + 1)^2$$

And

$$I_{min} \propto (A_1 - A_2)^2$$

$$I_{min} \propto A_2^2(\sqrt{\alpha} - 1)^2$$

$$\frac{I_{max} + I_{min}}{I_{max} - I_{min}} = \frac{A_2^2(\sqrt{\alpha} + 1)^2 + A_2^2(\sqrt{\alpha} - 1)^2}{A_2^2(\sqrt{\alpha} + 1)^2 - A_2^2(\sqrt{\alpha} - 1)^2}$$

$$\frac{I_{max} + I_{min}}{I_{max} - I_{min}} = \frac{(\sqrt{\alpha} + 1)^2 + (\sqrt{\alpha} - 1)^2}{(\sqrt{\alpha} + 1)^2 - (\sqrt{\alpha} - 1)^2}$$

$$\frac{I_{max} + I_{min}}{I_{max} - I_{min}} = \frac{\alpha + 1 + 2\sqrt{\alpha} + \alpha + 1 - 2\sqrt{\alpha}}{\alpha + 1 + 2\sqrt{\alpha} - \alpha - 1 + 2\sqrt{\alpha}}$$

$$\frac{I_{max} + I_{min}}{I_{max} - I_{min}} = \frac{2(\alpha + 1)}{4\sqrt{\alpha}} = \frac{\alpha + 1}{2\sqrt{\alpha}}$$

Young’s double slit experiment

The phenomenon of interference was first observed and demonstrated by Thomas Young in 1801. The experimental set up is shown in Fig.

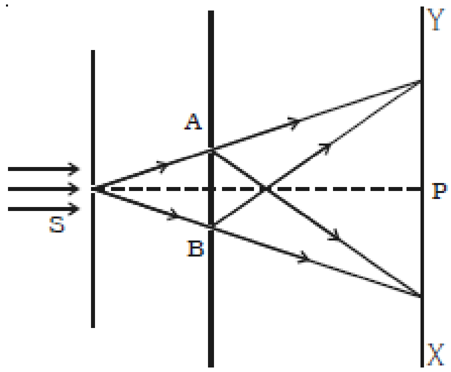
Light from a narrow slit S, illuminated by a monochromatic source, is allowed to fall on two narrow slits A and B placed very close to each other.

The width of each slit is about 0.03 mm and they are about 0.3 mm apart. Since A and B are equidistant from S, light waves from S reach A and B in phase. So A and B act as coherent sources.

According to Huygen’s principle, wavelets from A and B spread out and overlapping takes place to the right side of AB. When a screen XY is placed at a distance of about 1 metre from the slits, equally spaced alternate bright and dark fringes appear on the screen. These are called interference fringes or bands.

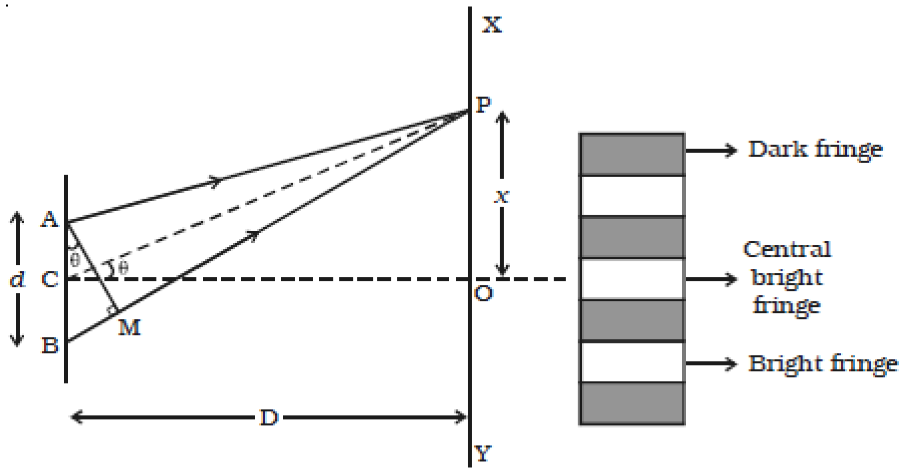
Using an eyepiece the fringes can be seen directly. At P on the screen, waves from A and B travel equal distances and arrive in phase. These two waves constructively interfere and bright fringe is observed at P. This is called central bright fringe.

When one of the slits is covered, the fringes disappear and there is uniform illumination on the screen. This shows clearly that the bands are due to interference.



Expression path difference in terms of D and x

Let *d* be the distance between two coherent sources A and B of wavelength λ. A screen XY is placed parallel to AB at a distance D from the coherent sources. C is the mid point of AB. O is a point on the screen equidistant from A and B. P is a point at a distance *x* from O, as shown in Fig. Waves from A and B meet at P in phase or out of phase depending upon the path difference between two waves.



Draw AM perpendicular to BP The path difference δ = BP – AP AP = MP

∴ δ = BP – AP = BP – MP = BM

In right angled Δ ABM, BM = d sin θ

If θ is small, sin θ = θ

∴ The path difference δ = θ.d

In right angled triangle COP,

$$\tan\theta = \frac{OP}{CO} = \frac{x}{D}$$

For small values of θ, tan θ = θ

∴ The path difference

$$\delta = \frac{xd}{D}$$

Location of bright and dark fringes on screen (x)

Bright fringes:

By the principle of interference, condition for constructive interference is the path difference = nλ

$$n\lambda = \frac{xd}{D}$$

$$x = \frac{n\lambda D}{d}$$

By substituting n = 0, 1, 2, 3…

If n = 0 then we get location central bright fringe

n=1, we get then location of first bright fringe

n=2, we get then location of second bright fringe

n=3, , we get then location of third bright fringe… etc

Dark fringes:

By the principle of interference, condition for destructive interference is the path difference

$$\delta = (2n-1)\frac{\lambda}{2}$$

$$(2n-1)\frac{\lambda}{2} = \frac{xd}{D}$$

$$x = \frac{D(2n-1)\lambda}{2d}$$

By substituting n = 1, 2, 3…

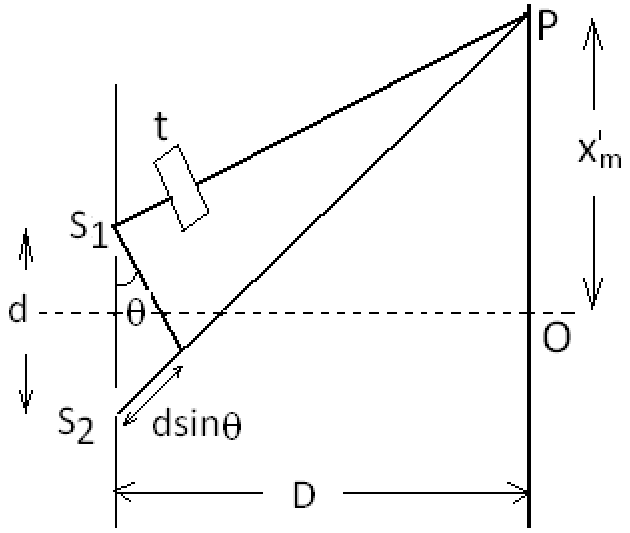
n=1, we get then location of first dark fringe

n=2, we get then location of second dark fringe

n=3, we get then location of third dark fringe… etc

Displacement of fringe pattern

If a thin transparent slab of thickness t and refractive index μ is placed in front of one of sources, for example, in front of S_1. This changes the path difference because light from S_1 now travels more optical path than earlier. *(Optical path in medium is equal to the product of refractive index of the medium to geometrical path length in air)*



Path difference before placing slab = $S_2P - S_1P = r_2 - r_1 = \delta$

On placing slab path length of thickness t effective path length $S_1P = (r_1 - t) + t\mu$

Thus effective path difference after placing slab $\delta' = r_2 - [(r_1 - t) + t\mu]$

$\therefore \delta' = (r_2 - r_1) + t(\mu - 1)$

From the geometry of figure $r_2 - r_1 = d\sin\theta$, since θ is very small

$$r_2 - r_1 = d\tan\theta = \frac{dx'_m}{D}$$

$$\therefore \delta' = \frac{dx'_m}{D} + (\mu - 1)t$$

$$x'_m = \frac{n\lambda D}{d} - (\mu - 1)\frac{tD}{d}$$

In absence of slab, the m^th maxima is given by

$$x_m = \frac{n\lambda D}{d}$$

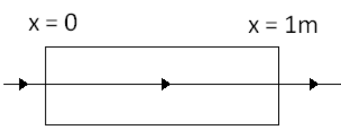
Therefore, the fringe shift is given by

$$x_0 = x_m - x'_m = (\mu - 1)\frac{tD}{d}$$

When a transparent slab is introduced, the fringe pattern shifts in the direction where the slab is placed.

Solved numerical

Q) A ray of light travels through a slab as shown in figure. The refractive index of the material of the slab varies as $\mu = 1.2 + x$, where $0 \le x \le 1$ m. What is the equivalent optical path of the glass slab?



Solution

Consider a small geometric path dx then optical path = μdx

Thus optical path op =

$$op = \int_0^1 (1.2 + x)dx = \left[1.2x + \frac{x^2}{2}\right]_0^1$$

$$op = 1.2 + \frac{1}{2} = \frac{3.4}{2} = 1.7$$

Distance between two consecutive bight or dark fringe
Or Width of fringe , Band width (β)

The distance between any two consecutive bright or dark bands is called bandwidth.
The distance between (n+1)th and nth order consecutive bright fringes from O is given by

$$\beta = x_{n+1} - x_n = (n+1)\frac{\lambda D}{d} - (n)\frac{\lambda D}{d} = \frac{\lambda D}{d}$$

Similarly, it can be proved that the distance between two consecutive dark bands is also equal to $\frac{\lambda D}{d}$

Angular fringe width or angular separation between fringes is

$$\theta = \frac{\lambda}{d}$$

Since bright and dark fringes are of same width, they are equi–spaced on either side of central maximum.

Solved numerical

Q) In Young's double slit experiment, angular width of a fringe formed on a distant screen is 0.1^{O}. the wavelength of the light used is 6000Å . What is the spacing between the slit. If above setup is immersed in liquid it is observed that angular fringe width is decreased by 30% find refractive index of liquid

Solution

Angular fringe width or angular separation between fringes is

$$\theta = \frac{\lambda}{d}$$

$$d = \frac{\lambda}{\theta} = \frac{6000 \times 10^{-10}}{0.1 \times \frac{\pi}{180}} = 3.44 \times 10^{-4} m$$

(ii)Given that when set up with first light is immersed in liquid angular width decreases by 30% thus wave length of first light in liquid = 4200Å

From formula for refractive index

$$\mu = \frac{\lambda_{air}}{\lambda_{liq}} = \frac{6000}{4200} = 1.428$$

Q) In Young's double slit experiment using monochromatic light the fringe pattern shifts by certain distance on the screen when a mica sheet of refractive index 1.6and thickness 1.964 micron is introduced in the path of one of the interfering waves. The mica sheet is then removed and the distance between the plane of the slit and the screen is doubled. It is found that the distance between successive maximum now is the same as the observed fringe shift upon the introduction of mica sheet. Calculate the wavelength of the light.

Solution

Due to introduction of mica sheet, the shift on the screen

$$x_0 = (\mu - 1)\frac{tD}{d}$$

Given that on removal of mica sheet and increasing the distance between screen and slit two times, then it is observed that distance between successive maximum now is the same as the *observed fringe shift* upon the introduction of mica sheet

Thus $x_o = \beta$ fringe width when distance is doubled

Now fringe width

$$\beta = \frac{\lambda(2D)}{d}$$

Therefore

$$(\mu - 1)\frac{tD}{d} = \frac{\lambda(2D)}{d}$$

$$\lambda = \frac{(\mu - 1)t}{2}$$

$$\lambda = \frac{(1.6 - 1)(1.964 \times 10^{-6})}{2} = 5892Å$$

Q) In Young's double slit experiment a beam of light of wavelength 6500Å and 5200Å is used . Find the minimum distance from the central bright fringe where bright fringe produced by both the wavelength get superposed. The distance between two slit is 0.5mm and the distance between the slits and the screen is 100cm.

Solution

Let n^{th} bright fringe due wavelength 6500Å to and m^{th} bright fringe due to 5200Å superposed at distance x from central bright spot

Thus $x_n = x_m$

$$\frac{n\lambda_1 D}{d} = \frac{m\lambda_2 D}{d}$$

$$n\lambda_1 = m\lambda_2$$

$$n6500 = m5200$$

$$\frac{m}{n} = \frac{6500}{5200} = \frac{5}{4}$$

That is 4^{th} bright fringe of 6500 Å and 3^{rd} bright fringe of 5200 Å superpose

Taking n =4 in equation we get the minimum distance from the central bright fringe where bright fringe produced by both the wavelength get superposed

$$x_n = \frac{n\lambda_1 D}{d}$$

$$x_n = \frac{4 \times 6500 \times 10^{-8} \times 100}{0.05} = 0.52cm$$

Q) The ratio of the intensities at minima to maxima in the interference pattern is 9:25. What will be the ratio of the widths of the two slits in young's double slit experiment

Solution

Intensity is proportional to width of slit, so amplitude is proportional to the square root of the width of the slit

$$\frac{A_1}{A_2} = \sqrt{\frac{w_1}{w_2}}$$

W_1 and w_2 represent the width of the two slits

$$\frac{I_{min}}{I_{max}} = \frac{(A_1 - A_2)^2}{(A_1 + A_2)^2}$$

$$\frac{I_{min}}{I_{max}} = \frac{\left(1 - \frac{A_2}{A_1}\right)^2}{\left(1 + \frac{A_2}{A_1}\right)^2}$$

$$\frac{9}{25} = \frac{\left(1 - \frac{A_2}{A_1}\right)^2}{\left(1 + \frac{A_2}{A_1}\right)^2}$$

$$\frac{3}{5} = \frac{1 - \frac{A_2}{A_1}}{1 + \frac{A_2}{A_1}}$$

$$8\frac{A_2}{A_1} = 2$$

$$\frac{A_1}{A_2} = \frac{4}{1}$$

Thus from equation (1)

$$\frac{4}{1} = \sqrt{\frac{w_1}{w_2}}$$

$$\frac{w_1}{w_2} = \frac{16}{1}$$

Condition for obtaining clear and broad interference bands

(i) The screen should be as far away from the source as possible.

(ii) The wavelength of light used must be larger.

(iii) The two coherent sources must be as close as possible.

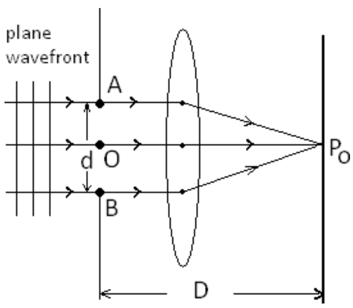
Diffraction

When waves encounters obstacles or openings like slits, they bend round the edges. This bending of wave is called diffraction. Diffraction is the effect produced by the limiting part of the wavefront.

Smaller is the width of the slit, more will be diffraction for given wavelength. It is also found that if the wavelength and the width of the slit are so changed that ratio (λ/d) remains constant, amount of bending or diffraction does not change.

If ratio λ/d is more ,then more is the diffraction

Diffraction due to single slit



Central Maxima:

Consider a plane wavefront arrive at a plane of slit, according to Huyen's principle all the point on the slit like AOB acts as secondary source having the same phase and produce secondary waves.

Those waves originated from each points of a slit and diffracted normal to the plane of the slit or we can say in the direction of incident wave will be concentrated at point P_o by lens.

In figure out of a many waves only three rays are shown in figure.

Screen is at focal length of lens .

Ray emitted from A and B are in phase and passes equal distance through air and lens thus they are in phase when get converged at P_o .

Now ray emitted from O travel less distance in air but more distance in lens, in lens velocity of light gets reduced thus optical path travelled by the ray emitted by O is equal to optical path due to ray A and B.. Thus all rays meeting at P^o are in phase produces central bright fringe.

(Optical path in medium is equal to the product of refractive index of the medium to geometrical path length in air)

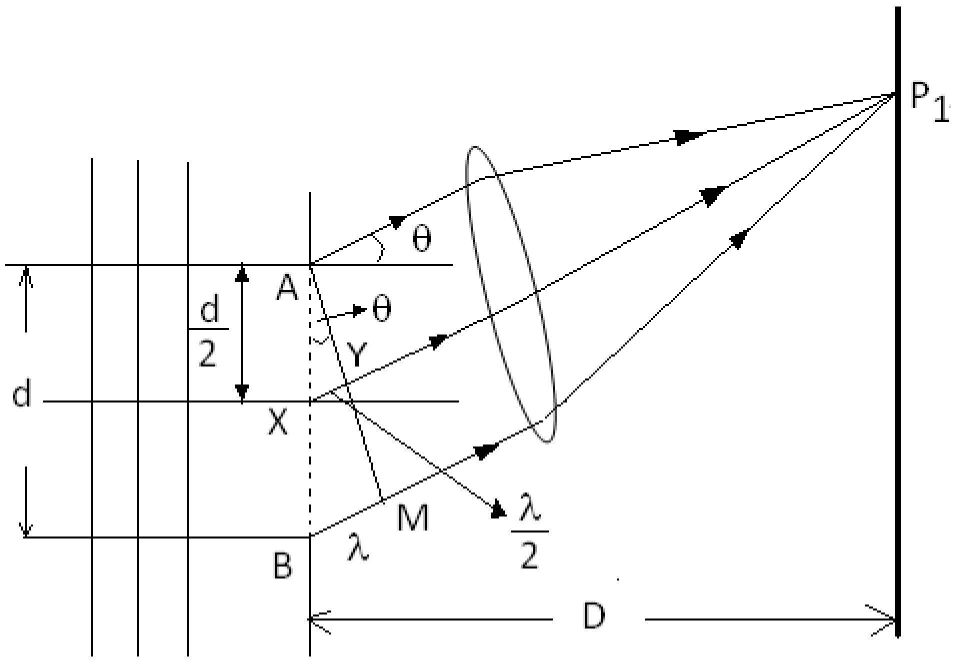
First minimum

As shown in figure consider a waves which is diffracted an angle θ with respect to perpendicular bisector XP_o of the slit. Here, point X is the midpoint of slit AB. Therefore AX = Xb = d/2.

Here secondary waves originated from all points A, X, B of slit are through to be divided in two parts

Wave from AX and waves from X to B.

As per figure, all these waves diffracted at an angle θ are focused at point P_1 of a screen. Draw AM $\perp$ BL. It is obvious that all the rays reaching from AM to P_1 have equal optical path

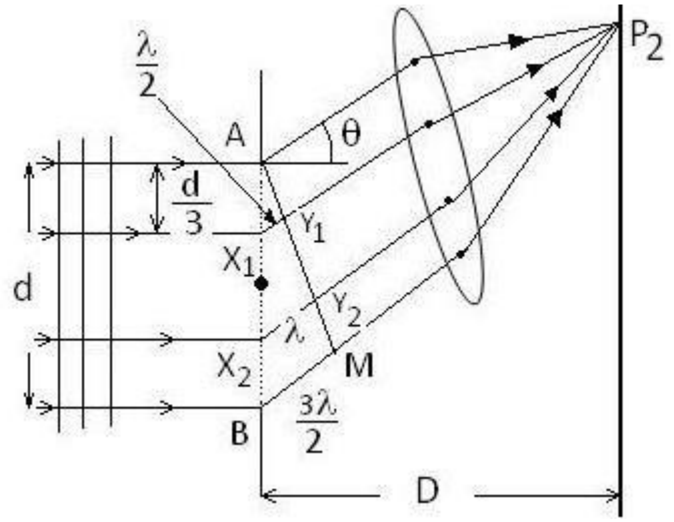


But rays going from A and X, and reaching to point P_1 have path difference of XY
Let assume diffraction angle be θ is such that XY = λ/2
In this situation, waves from A and X will follow the condition of destructive interference at point P_1 and resultant intensity will be zero
Further for all point between AX there exists a point between XB ,such that ray from point between XB have path difference of λ/2 with respect to rays from point between AX
Thus in totality, destructive interference will take place at point P_1
Point P_1 is known as first minimum

Condition for minima:

From geometry of figure dsinθ = λ
General equation is
dsinθ = nλ
For n = 1 we get first minima
n = 2 we get second minima

First Maxima:
As shown in figure suppose slit AB is assumed to be divided in three equal parts AX_1, $X_1 X_2$, X_2B . Here $AX_1 = X_1 X_2 = X_2B = d/3$.
Draw AM ⊥ BL. Wave reaching from AM to P_2 will have equal optical path
Waves starting from A and X_1 and imposing at point P_1 will have path difference X_1Y_1 .



Let us assume that diffraction θ is such that

$$X_1Y_1 = \frac{\lambda}{2}$$

$$X_2Y_2 = \lambda$$

$$BM = \frac{3\lambda}{2}$$

Since path difference between waves originated from A and X_1 and superimpose at point P_2 is λ/2, they interfere destructively. And intensity at point P_2 due to these waves will be zero.

In the same way, waves from every pair AX_1 and X_1X_2 will have path difference λ/2 and resultant intensity at point P_2 due to them is zero.

However, intensity of ray diffracted at an angle θ from section X_1B is not vanishing at point P_2. Therefore due to this section of the slit intensity at point P_2 will not be zero. And point P_2 will be bright

Here point P_2 is known as first maximum. It is obvious that intensity at point P_2 will be far less than central bright spot

Condition for minima:

From geometry of figure for fist maxima

$$d\sin\theta = \frac{3\lambda}{2}$$

General formula

$$d\sin\theta = \frac{(2n+1)\lambda}{2}$$

For n = 1 we get first maxima

n = 2 we get second maxima

Angular width

For first order minima dsinθ =λ or

$$sin\theta = \frac{\lambda}{d}$$

For small angle

$$\theta = \frac{\lambda}{d}$$

Also sinθ = tanθ

From geometry of figure

$$tan\theta = \frac{x}{D}$$

$$\therefore \frac{x}{D} = \frac{\lambda}{d}$$

∴ width of central maxima

$$2x = \frac{2\lambda D}{d}$$

Angular width of central maxima is given by

$$2\theta = \frac{2\lambda}{d}$$

Comparison between Interference and diffraction

(i) The interference pattern has a number of equally spaced bright and dark bands. The diffraction pattern has a central bright maximum which is twice as wide as the other maxima. The intensity falls as we go to successive maxima away from the centre, on either side.

(ii) We calculate the interference pattern by superposing two waves originating from the two narrow slits. The diffraction pattern is a superposition of a continuous family of waves originating from each point on a single slit.

(iii) For a single slit of width *a*, the first null of the interference pattern occurs at an angle of λ/*a.* At the same angle of λ/*a*, we get a maximum (not a null) for two narrow slits separated by a distance *a.*

Solved numerical

Q)Angular width of central maximum in diffraction obtained by single slit using light of wavelength 6000Å is measured. If light of another wavelength is used, the angular width of the central maximum is found to be decreased by 30%. Find

(i)The other wavelength (ii) If the experiment is repeated keeping the apparatus in a liquid, the angular width of central maxima decreases by the same amount (30%0, find its refractive index

Solution:

(i)Angular fringe width or angular separation between fringes is

$$2\theta = \frac{2\lambda}{d}$$

For first light

$$\theta_1 = \frac{\lambda_1}{d}$$

For second light

$$\theta_2 = \frac{\lambda_2}{d}$$

$$\frac{\theta_2}{\theta_1} = \frac{\lambda_2}{\lambda_1}$$

But θ_2 is 70% of θ_1
That is , $\theta_2 = 0.7\theta_1$

$$\therefore 0.7 = \frac{\lambda_2}{\lambda_1}$$

$$\lambda_2 = 0.7 \times 6000 = 4200 \text{ Å}$$

Q) A slit of width d is illuminated by white light. For what value of d will the first minimum for red light of wavelength λ_R =6500Å appear at θ =15^O? What is the situation for violet colour having wavelength λ_V = 4333Å at the same point. Sin15^O = 0.2588
Solution:
Since the diffraction occurs separately for each wavelength, we have to check condition for minia and maxima for each wavelength separately
For the first minimum of red colour n = 1, using condition
$d\sin\theta = n\lambda_R$

$$d = \frac{n\lambda_R}{sin\theta} = \frac{1 \times 6500 \times 10^{-10}}{sin 15}$$

$d=2.512\times10^{-6}$ m
For violet colour wavelength is different. We have to check whether slit width above can give us minima or maxima
Thus $d\sin\theta = n'\lambda_V$

$$n' = \frac{dsin\theta}{\lambda_v} = \frac{2.512 \times 10^{-6} \times 0.2588}{4333 \times 10^{-10}} = 1.5$$

Here n' should be integer. Thus, for violet colour condition for minima does not satisfy
Now using condition for maxima

$$dsin\theta = (2n' + 1)\frac{\lambda_V}{2}$$

$$n' = \frac{dsin\theta}{\lambda_v} - \frac{1}{2} = 1.5 - 0.5 = 1.0$$

This result suggest that for violet colour first maximum is observed

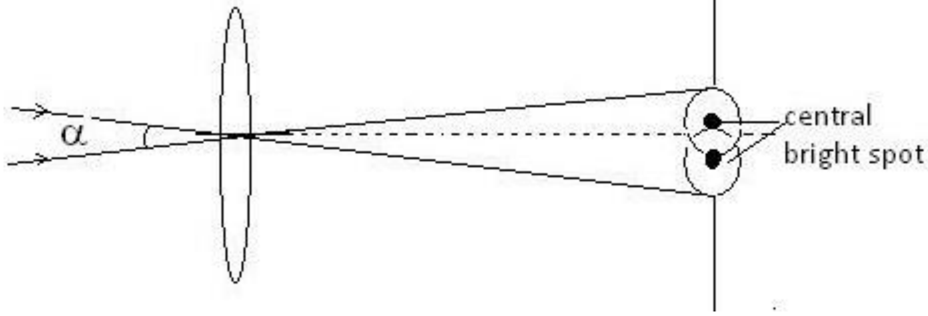
Resolving power of optical instrument

When a beam of light (light waves) from a point like object passes through the objective of an optical instruments, the lens acts like a circular aperture and produces a diffraction pattern instead of sharp point image.
If there are two point objects kept closed to each other, their diffraction pattern may overlap. Then it may be difficult to distinguish them as separate.
The criterion to get distinct and separate images of two closely placed point like objects was given by Rayleigh
" The images of two point like objects can be seen as separate if the central maximum in the diffraction pattern of one falls either on the first minimum of the diffraction pattern of the other or it is at grater separation"
For the case of circular aperture diffraction due to lens of diameter D. Rayleigh's criterion is given by

$$sin\alpha \approx \alpha = \frac{1.22\lambda}{D}$$

Resolving power of telescope:

Consider a parallel beam of light falling on a convex lens. If the lens is well corrected for aberrations, then beam will get focused to a point.
However, because of diffraction, the beam instead of getting focused to a point gets focused to a spot of finite area. In this case the effects due to diffraction can be taken into account by considering a plane wave incident on a circular aperture followed by a convex lens.
Taking into account the effects due to diffraction, the pattern on the focal plane would consist of a central bright region surrounded by concentric dark and bright rings.



If two stars are very close to each other separated by angle α will be very small and the diffraction pattern of both stars will mingle with each other. In this situation it is difficult to see both the stars distinctly and clearly

"Ability of an optical instrument to produce distinctly separate images of two closely placed objects is called its resolving power"

It is clear that for optical instruments resolving power depends on angle α . is a minimum angle to see two images distinctly

$$\alpha_{min} = \frac{1.22\lambda}{D}$$

Here D is diameter of lens and λ is wavelength

Width of the central maxima or radius is given by

$$\alpha_{min} f = \frac{1.22\lambda}{D} f$$

Here α_{min} is known as angular resolution of the telescope, while its inverse is known as resolving power or geometrical resolution

Thus resolving power of telescope

$$\frac{1}{\alpha_{min}} = \frac{D}{1.22\lambda}$$

Solved numerical

Q) Calculate the useful magnifying power of a telescope of 11cm objective. The limit of resolution of eye is 2' and wavelength of light used is 5000Å

Solution

The magnifying power of a telescope is given by

M = D/d, where D is diameter of the objective and d is diameter of eye piece

For normal (useful) magnification, diameter of eyepiece should be equal to the diameter of the pupil d_e of the eye. Therefore, useful magnification is

M = D/d_e

From the equation of limit of resolution of telescope

$$d\theta = \frac{1.22\lambda}{D}$$

$$d\theta = \frac{1.22 \times 5500 \times 10^{-10}}{11 \times 10^{-2}} = 6.1 \times 10^{-6}\, rad$$

Limit of resolution of eye is given dθ'=2'

$$d\theta' = \frac{2 \times 3.14}{60 \times 180^o} = 5.815 \times 10^{-4} rad$$

∴ Useful magnification

$$\frac{d\theta'}{d\theta} = \frac{5.815 \times 10^{-4}}{6.1 \times 10^{-6}} = 95.3$$

Q) Huble space telescope is at a distance 600 km from earth's surface. Diameter of its primary lens (objective) is 2.4m. When a light of 550nm is used by this telescope, at what minimum angular distance two objects can be seen separately? Also obtain linear minimum distance between these objects. Consider these objects on the surface of earth and neglect effects of atmosphere.

Solution:

$$\alpha_{min} = \frac{1.22\lambda}{D} = \frac{1.22 \times 550 \times 10^{-9}}{2.4}$$

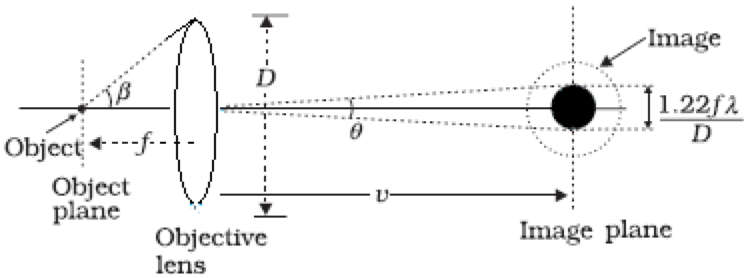
$$\alpha_{min} = 2.8 \times 10^{-7} rad$$

Linear distance between objects= $\alpha_{min}L$

Where L = distance between object telescope and object

Linear distance between objects = $2.8 \times 10^{-7} \times 600 \times 10^{3}$ = 0.17m

Resolving power of microscope:



Let the diameter of lens be D and its focal length be f. As object distance is usually kept greater than that of f . Let the image distance be v. the angular width of central maximum due to the effect of diffraction is ,

$$\theta = \frac{1.22\lambda}{D}$$

Width of central maximum

$$\theta v = \frac{1.22\lambda}{D} v$$

If image of two point like objects are at a separation les than vθ, then it will be seen as a mixed single object. It can be proved that a minimum distance (d_m) for which objects can be seen separately is given by

$$d_m = \theta \frac{v}{m}$$

Here m is magnification m=v/f substituting value of m we get d_m

$$d_m = \theta \frac{v}{v/f} = \theta f$$

Substituting value of θ we get

$$d_m = \frac{1.22\lambda}{D} f$$

From figure D/2 =f (tanβ)

D = 2f (tanβ) substituting value in above equation we get

$$d_m = \frac{1.22\lambda}{2f tan\beta} f = \frac{1.22\lambda}{2tan\beta}$$

For small angles tanβ = sinβ

Reciprocal of d_m known as Resolving Power(RP) of microscope

$$RP = \frac{1}{d_m} = \frac{2\sin\beta}{1.22\lambda}$$

If some medium with large refractive index (n) is used between object and objective resolving power of microscope increases n times
Formula for resolving power is given by

$$RP = \frac{2n\sin\beta}{1.22\lambda}$$

Here term nsinβ is known as 'Numerical Aperture". Resolving power is inversely proportional to wavelength.

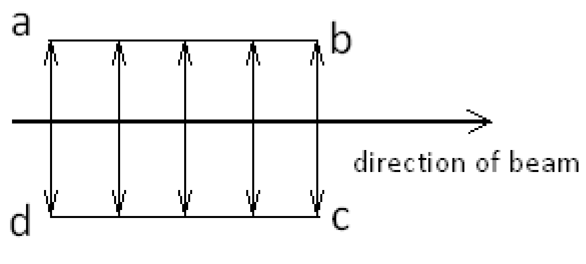
Polarization

The phenomena of reflection, refraction, interference, diffraction are common to both transverse waves and longitudinal waves. But the transverse nature of light waves is demonstrated only by the phenomenon of polarization.

Unpolarized light
In an ordinary light source like bulb, there are large numbers of atomic emitters. They all emit electromagnetic waves with there Electrical vector E , vibrating randomly in all directions perpendicular to direction of propagation.
It means that vector E of one wave is not parallel to Vector E of another wave.
Wave emitted by different atom is of source propagate in same direction constitute beam of light.
If such beam is assumed to be coming out of paper, light vectors (E) of its waves will be found in all random direction in a plane of paper. Such light is called Unpolarized light.
" In a beam of light, if the oscillations of E vectors are in all direction in a plane perpendicular to the direction of propagation, then the light is called unpolarized light"

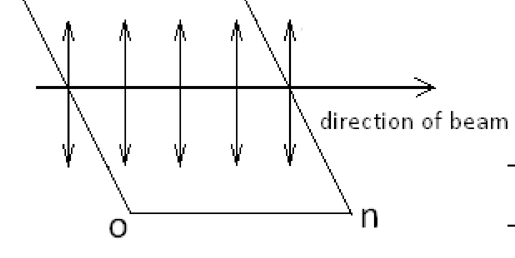
Polarized light

If in beam of light all electric vector (E) are coplanar and parallel to each other is plane polarized light
Process by which getting the plane polarized light from unpolarized light is called polarization



" The plane containing the direction of the beam and the direction of oscillation of E vectors is called the plane of oscillation . In figure abcd is the plane of oscillation

"A plane perpendicular to the plane of oscillation and passing through the beam of light is called the plane of polarization"

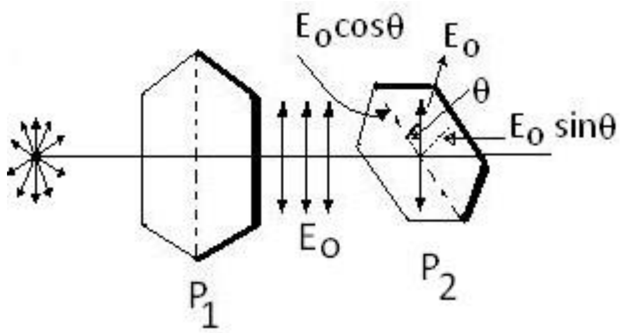


In figure lmno is the plane of polarization

When light passes through tourmaline crystal ,freely transmit the light components which are polarized to a definite direction. While crystal absorbs light strongly whose polarization is perpendicular to this definite direction. Thus emergent beam iof light only coplanar and parallel E vectors are found. This definite direction in a crystal is known as an *optic axis*

Malus' Law

If the light from an ordinary source (like a sodium lamp) passes through a polaroid sheet P_1, it is observed that its intensity is reduced by half. Rotating P_1 has no effect on the transmitted beam and transmitted intensity remains constant.
Now, let an identical piece of polaroid P_2 be placed before P_1. On rotating P_2 has a dramatic effect on the light coming from P_2.
In one position, the intensity transmitted by P_2 followed by P_1 is nearly zero. When turned by 90^o from this position, P_1 transmits nearly the full intensity emerging from P_2
An optic axis of plate P_2 makes an angle of θ with that of the plate P_1. In this situation vector E emerging from plate P_1 (E_0) makes angle θ with an optic axis of plate B. therefore we can resolve them into two components
91) $E_0\cos\theta$ parallel to the optic axis of plate P_2 and
(2) $E_0\sin\theta$ perpendicular to the optic axis of plate P_2



Thus, only $E_0\cos\theta$ components will emerge out of plate $P_{2'}$ while perpendicular components are absorbed. Since intensity is proportional to the square of amplitude, intensity of light incident on plate P_2 is

$$I \propto E_0^2 cos^2\theta$$
$$\therefore \frac{I}{I_0} = cos^2\theta$$
$$\therefore I = I_0 cos^2\theta$$

This equation is known as Malus Law. It is obvious from above equation that if plate P_2 is completely rotated, twice the intensity of emerging light is zero , corresponding to $\theta = \pi/2$ and $3\pi/2$ and twice it become maximum corresponding to $\theta = 0$ and $\theta = \pi$.
This procedure will help us to verify whether the given light is polarized or not. Since plate P_2 is used to analyze a state of polarization of incident light, it is known as Analyzer.

Solved numerical

Q) A ray of light travelling in water is incident on a glass plate immersed in it. What the angle of incident is 51° the reflected ray is totally plane polarized. Find the refractive index of glass. Refractive index of water is 1.33

Solution:

Angle of incidence $\theta_P = 51^o$

Since at this incidence angle, reflected ray is totally plane polarized, using Brewster'slaw refractive index of glass w.r.t. water is

n' = tan θ_P = tan51 = 1.235

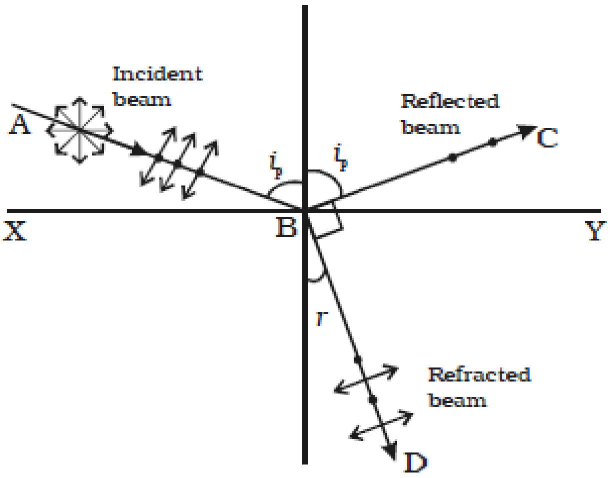
But refractive index n' =

$$n' = \frac{RI.\,glass(n_g)}{R.i.\,of\ water(n_w)}$$

$n_g = n'\, n_w = 1.235 \times 1.33 = 1.64$

Polarisation by reflection

The simplest method of producing plane polarised light is by reflection. Malus, discovered that when a beam of ordinary light is reflected from the surface of transparent medium like glass or water, it gets polarised. The degree of polarisation varies with angle of incidence.
Consider a beam of unpolarised light AB, incident at any angle on the reflecting glass surface XY. Vibrations in AB which are parallel to the plane of the diagram are shown by arrows. The vibrations which are perpendicular to the plane of the diagram and parallel to the reflecting surface, shown by dots (Fig).



A part of the light is reflected along BC, and the rest is refracted along BD.
On examining the reflected beam with an analyzer, it is found that the ray is partially plane polarised. When the light is allowed to be incident at a particular angle, (for glass it is 57.5°) the reflected beam is completely plane polarised. The angle of incidence at which the reflected beam is completely plane polarised is called the polarising angle (i_p).

Brewster's law

Sir David Brewster conducted a series of experiments with different reflectors and found a simple relation between the angle of polarization and the refractive index of the medium.
It has been observed experimentally that the reflected and refracted rays are at right angles to each other, when the light is incident at polarizing angle.
From Fig, $ip + 90^0 + r = 180^0$
$r = 900 - ip$
From Snell's law,

$$\frac{sini_p}{sinr} = \mu$$

where μ is the refractive index of the medium (glass)
Substituting for *r*, we get

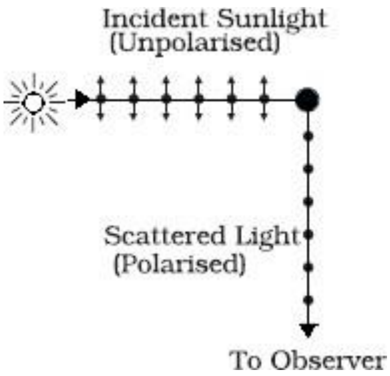
$$\frac{sini_p}{sin(90 - i_p)} = \frac{sini_p}{cosi_p} = \mu$$

$$\therefore \tan i_p = \mu$$

Tangent of polarizing angle is numerically equal to refractive index of medium

Polarisation by scattering

The light from a clear blue portion of the sky shows a rise and fall of intensity when viewed through a Polaroid which is rotated. This is nothing but sunlight, which has changed its direction (having been scattered) on encountering the molecules of the earth's atmosphere



As shown in figure, the incident sunlight is unpolarised. The dots stand for polarisation perpendicular to the plane of the figure. The double arrows show polarisation in the plane of the figure. There is no phase relation between these two in unpolarised light.
Under the influence of the electric field of the incident wave the electrons in the molecules acquire components of motion in both these directions.
We have drawn an observer looking at 90° to the direction of the sun. Clearly, charges accelerating parallel to the double arrows do not radiate energy towards this observer since their acceleration has no transverse component. The radiation scattered by the molecule is therefore represented by dots. It is polarized perpendicular to the plane of the figure. This explains the polarization of scattered light from the sky.

Fresnel distance, ray optics is a limiting case of wave optics

Fresnel distance is that distance from the slit at which the spreading of light due to diffraction becomes equal to the size of the slit. It is generally denoted by Z_F
We know that the first secondary minimum is formed at an angle θ_1 such that

$$\theta_1 = \frac{\lambda}{d}$$

After travelling a distance D, the width acquired by the beam due to diffraction is $D\lambda/d$
At Fresenel distance Z_F

$$\frac{Z_F\lambda}{d} = d$$

$$Z_F = \frac{d^2}{\lambda}$$

If the distance D between the slit and the screen is less than Fresnel distance Z_F then the diffraction effects may be regarded as absent. So, ray optics may be regarded as limiting case of wave optics

Solved Numerical

Light of wave length 600nm is incident on an aperture of size 2mm. calculate the distance upto which the ray of light can travel such that its spread is less than the size of the aperture
Solution

$$Z_F = \frac{d^2}{\lambda} = \frac{(2 \times 10^{-3})^2}{600 \times 10^{-9}} = 6.67\ m$$

Doppler effect for light

If there is no medium and the source moves away from the observer, then later wavefronts have to travel a greater distance to reach the observer and hence take a longer time. The time taken between the arrival of two successive wavefronts is hence longer at the observer than it is at the source.

Thus, when the source moves away from the observer the frequency as measured by the source will be smaller. This is known as the *Doppler effect*.

Astronomers call the increase in wavelength due to doppler effect as *red shift* since a wavelength in the middle of the visible region of the spectrum moves towards the red end of the spectrum.

When waves are received from a source moving towards the observer, there is an apparent decrease in wavelength, this is referred to as *blue shift.*

For velocities small compared to the speed of light, we can use the same formulae which we use for sound waves. The fractional change in frequency Δv/v is given by $-V_{radial}/c$, where V_{radial} is the component of the source velocity along the line joining the observer to the source relative to the observer; V_{radial} is considered positive when the source moves away from the observer. Thus, the Doppler shift can be expressed as:

$$\frac{\Delta v}{v} = -\frac{V_{radial}}{c}$$

Solved numerical

Q) Certain characteristic wavelengths of the light from a galaxy in the constellation Virgo are observed to be increased in wave length, as compaired with terrestrial sources, by 0.4%. What is the radial speed of this galaxy with respect to the earth? Is it approaching or receding?

Solution

From formula

$$\frac{\Delta v}{v} = -\frac{V_{radial}}{c}$$

We know that

$$\frac{\Delta v}{v} = -\frac{\Delta\lambda}{\lambda}$$

Thus

$$\frac{\Delta\lambda}{\lambda} = \frac{V_{radial}}{c}$$

Given : Δλ/λ = 0.004

$$V_{radial} = \frac{\Delta\lambda}{\lambda} c$$

$$V_{radial} = 0.004 \times 3 \times 10^8 = 1.2 \times 10^6\ ms^{-1}$$

Since v_{radial} is positive therefore galaxy is receding

Q) the red shift of radiation from a distant nebula consists of the light known to have a wavelength 4340×10^{-8} cm when observed in laboratory, appearing to have a wavelength of 4362×10^{-8} cm. What is the speed of the nebula in the line of sight relative to the earth ? Is it approaching or receding

Solution:

$\Delta\lambda = 4362\times10^{-8} - 4340\times10^{-8} = 22\times10^{-8}$ cm thus

$\Delta\lambda/\lambda = 22\times10^{-8} /4340\times10^{-8} =0.0004$

Thus $\Delta\nu/\nu =$ -0.0004

$$\frac{\Delta v}{v} = -\frac{V_{radial}}{c}$$

$$-0.0004 = -\frac{V_{radial}}{3\times10^{8}}$$

$V_{radial} = (0.0004)(3\times10^{8}) = 0.12852\times10^{8} = 1.2\times10^{5}$ m/s

Since V_{radial} is positive nebula is receding

SOUND WAVES

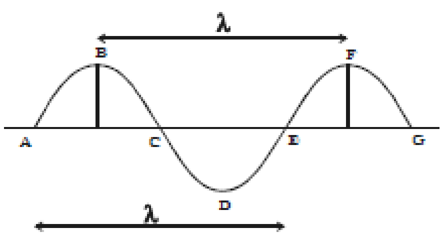
Wave motion is a mode of transmission of energy through a medium in the form of a disturbance. It is due to the repeated periodic motion of the particles of the medium about an equilibrium position transferring the energy from one particle to another.
The waves are of three types - mechanical, electromagnetic and matter waves. Mechanical waves can be produced only in media which possess elasticity and inertia. Water waves, sound waves and seismic waves are common examples of this type. Electromagnetic waves do not require any material medium for propagation. Radio waves, microwaves, infrared rays, visible light, the ultraviolet rays, *X* rays and γ rays are electromagnetic waves. The waves associated with particles like electrons, protons and fundamental particles in motion are matter waves.

Characteristics of wave motion

(i) Wave motion is a form of disturbance travelling in the medium due to the periodic motion of the particles about their mean position
(ii) It is necessary that the medium should possess elasticity and inertia.
(iii) All the particles of the medium do not receive the disturbance at the same instant (i.e) each particle begins to vibrate a little later than its predecessor.
(iv) The wave velocity is different from the particle velocity. The velocity of a wave is constant for a given medium, whereas the velocity of the particles goes on changing and it becomes maximum in their mean position and zero in their extreme positions.
(v) During the propagation of wave motion, there is transfer of energy from one particle to another without any actual transfer of the particles of the medium.
(vi) The waves undergo reflection, refraction, diffraction and interference

Mechanical wave motion

The two types of mechanical wave motion are (i) transverse wave motion and
(ii) longitudinal wave motion



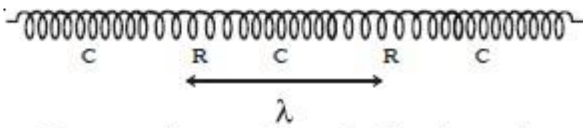
(i) Transverse wave motion

Transverse wave motion is that wave motion in which particles of the medium execute SHM about their mean positions in a direction perpendicular to the direction of propagation of the wave. Such waves are called transverse waves. Examples of transverse waves are waves produced by plucked strings of veena, sitar or violin and electromagnetic waves. Transverse waves travel in the form of crests and troughs. The maximum displacement of the particle in the positive direction i.e. above its mean position is called

crest and maximum displacement of the particle in the negative direction i.e below its mean position is called trough.
Thus if ABCDEFG is a transverse wave, the points B and F are crests while D is trough For the propagation of transverse waves, the medium must possess force of cohesion and volume elasticity.
Since gases and liquids do not have rigidity (cohesion), transverse waves cannot be produced in gases and liquids. *Transverse waves can be produced in solids and surfaces of liquids only*

***(ii)* Longitudinal wave motion**

'Longitudinal wave motion is that wave motion in which each particle of the medium executes simple harmonic motion about its mean position along the direction of propagation of the wave.'
Sound waves in fluids (liquids and gases) are examples of longitudinal wave. When a longitudinal wave travels through a medium, it produces compressions and rarefactions.
In the case of a spiral spring, whose one end is tied to a hook of a wall and the other end is moved forward and backward, the coils of the spring vibrate about their original position along the length of the spring and longitudinal waves propagate through the spring



Compression and rarefaction in spring

The regions where the coils are closer are said to be in the state of compression, while the regions where the coils are farther are said to be in the state of rarefaction.
When sound waves passes through that region of air, the air molecules in certain region are pushed very close to each other during their oscillations. Hence, both density and pressure of air increase in such regions. In such region condensation is said to be formed.
In the regions between two consecutive condensations, the air molecules are found to be quit separated. In such region density and pressure of air decreases and rarefaction is said to be formed.
Compressive strain is produced during the propagation of waves, which is possible in solid, liquids and gases medium.

Important terms used in wave motion

(i) Wavelength (λ)

The distance travelled by a wave during which a particle of the medium completes one vibration is called wavelength. It is also defined as the distance between any two nearest particles on the wave having same phase.
Wavelength may also be defined as the distance between two successive crests or troughs in transverse waves, or the distance between two successive compressions or rarefactions in longitudinal waves.

(ii) Time period (T)
The time period of a wave is the time taken by the wave to travel a distance equal to its wavelength.
(iii) Frequency (n)
This is defined as the number of waves produced in one second. If T represents the time required by a particle to complete one vibration, then it makes 1/T*waves in one second.*
Therefore frequency is the reciprocal of the time period
(i.e) $F = 1/T$

Relationship between velocity, frequency and wavelength of a wave
The distance travelled by a wave in a medium in one second is called the velocity of propagation of the wave in that medium. If v represents the velocity of propagation of the wave, it is given by

$$Velocity = \frac{Distance}{time}$$

$$v = \frac{\lambda}{T} = \lambda f$$

The velocity of a wave (v) is given by the product of the frequency and wavelength.

Velocity of wave in different media
The velocity of mechanical wave depends on elasticity and inertia of the medium
Velocity of a transverse wave along a stretched string
If m is the mass per unit length of the string
If T is the tension in string. Then velocity of wave

$$v = \sqrt{\frac{T}{m}}$$

The speed of a wave along a stretched ideal string depends only on the tension and the linear mass density of the string and does not depend on the frequency of the wave

Velocity of longitudinal waves in an elastic medium
Velocity of longitudinal waves in an elastic medium is

$$v = \sqrt{\frac{E}{\rho}}$$

where E is the modulus of elasticity, ρ is the density of the medium.

(i) In the case of a solid rod

$$v = \sqrt{\frac{Y}{\rho}}$$

where *Y* is the Young's modulus of the material of the rod and ρ is the density of the rod.
(ii) In liquids,

$$v = \sqrt{\frac{B}{\rho}}$$

where *B* is the Bulk modulus and ρ is the density of the liquid

Newton's formula for the velocity of sound waves in air

Newton assumed that sound waves travel through air under isothermal conditions (i.e) temperature of the medium remains constant.
The change in pressure and volume obeys Boyle's law.
PV = constant
Differentiating we get
P dV + V dP = constant

$$\therefore P - V\frac{dP}{dV} = \frac{dP}{dV/V} = Bulk\ Modulus\ B$$

Thus, isothermal bulk modulus B = Pressure P

$$\therefore v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{P}{\rho}}$$

Since solids and liquids are much less compressible than gases, speed of sound in gasses is higher.

Laplace's correction

The experimental value for the velocity of sound in air is 332 m s^{-1}. But the theoretical value of 280 m $^{s-1}$ is 15% less than the experimental value.
The above discrepancy between the observed and calculated values was explained by Laplace in 1816.
Sound travels in air as a longitudinal wave. The wave motion is therefore, accompanied by compressions and rarefactions. At compressions the temperature of air rises and at rarefactions, due to expansion, the temperature decreases. Air is a very poor conductor of heat. Hence at a compression, air cannot lose heat due to radiation and conduction. At a rarefaction it cannot gain heat, during the small interval of time. As a result, the temperature throughout the medium does not remain constant. Laplace suggested that sound waves travel in air under adiabatic condition and not under isothermal condition.
For an adiabatic change, the relation between pressure and volume is given by

$$PV^{\gamma} = constant$$

Differentiating the equation with respect to V

$$P\,\gamma V^{\gamma-1} + V^{\gamma}\frac{dP}{dV} = 0$$
$$P\,\gamma + V\frac{dP}{dV} = 0$$
$$\frac{-dP}{dV/V} = \gamma P$$
$$\therefore B = \gamma P$$

Thus, for an adiabatic process bulk modulus = γP

Using this value of B we get wave speed

$$v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{\gamma P}{\rho}}$$

Factors affecting velocity of sound in gases

(i) Effect of pressure

If pressure of the gas is changed keeping its temperature constant , P/ρ remains constant as the density of gas directly varies as the pressure. Therefore, the speed of sound in a gas does not depend on the pressure of the gas, at constant temperature and constant humidity

Density of water vapour is less than the density of dry air at same temperature. Hence the speed of sound increases with increase in humidity as equation

$$v = \sqrt{\frac{\gamma P}{\rho}}$$

(ii) Effect of temperature

For a gas, *PV = RT for one mole of gas*

$$P = \frac{RT}{V}$$

Substituting value of P in equation

$$v = \sqrt{\frac{\gamma P}{\rho}}$$

We get

$$v = \sqrt{\frac{\gamma}{\rho}\frac{RT}{V}} = \sqrt{\frac{\gamma RT}{m}}$$

Mass of gas is $m = \rho V$

Speed of sound in gas is directly proportional to the square root of its absolute temperature (T)

$$v \propto \sqrt{T}$$

velocity of sound in air increases by 0.61 m s^{-1} per degree centigrade rise in temperature

(iii) ***Effect of wind***

The velocity of sound in air is affected by wind. If the wind blows with the velocity *w* along the direction of sound, then the velocity of sound increases to *v + w*. If the wind blows in the opposite direction to the direction of sound, then the velocity of sound decreases to *v – w*. If the wind blows at an angle θ with the direction of sound, the effective velocity of sound will be (*v + w* cos θ).

Note: In a medium, sound waves of different frequencies or wavelengths travel with the same velocity. Hence there is no effect of frequency on the velocity of sound.

Solved Numerical

Q) The wavelength of a note emitted by a tuning fork of frequency 512Hz in air at $17^{O}C$ is 66.5 cm. If the density of air at S.T.P. is 1.293 g/lit, calculate γ of air.

Solution:

Frequency of tuning fork = 512 Hz, T = 17+28-73 = 290K, λ = 0.665 m

Density of air = 1.293 g/litre = 1.293 kg/m^3 pressure P = 1 atm = 1.01×10^5 N/m^2

Velocity of sound v = fλ = 512 × 0.665 = 340.5 m/s

$$v = \sqrt{\frac{\gamma P}{\rho}}$$

$$\gamma = \frac{v^2\rho}{P}$$

$$\gamma = \frac{(340.5)^2 \times 1.293}{1.01 \times 10^5} = 1.48$$

Q) The speed of transverse wave going on a wire having length 50cm and mass 5.0g is 80 m/s. The area of cross-section of the wire is 1.0 mm^2 and its Young's modulus is 16×10^{11}N/m^2. Find the extension of the wire over its natural length

Solution:

Length of wire L = 50 cm = 50×10^{-2} m

Mass of wire M = 5g = 5×10^{-3}kg

Cross sectional area of wire A = 1mm^2 = 10^{-6} m^2

Young's modulus of wire Y = 16×10^{11}N/m^2

Mass per unit length of wire m = M/L = 5×10^{-3}kg/50×10^{-2} m =10^{-2} kg/m

The wave speed in wire

$$v = \sqrt{\frac{T}{m}}$$

$$\therefore T = mv^2$$

$$\therefore T = 10^{-2} \times (80)^2 = 64N$$

Now Young's modulus

$$Y = \frac{F/_A}{\Delta L/_L}$$

Extension of wire ΔL

$$\Delta L = \frac{FL}{AY}$$

$$\Delta L = \frac{(64)(50 \times 10^{-2})}{(10^{-6})(16 \times 10^{11})} = 0.02\ mm$$

Progressive wave

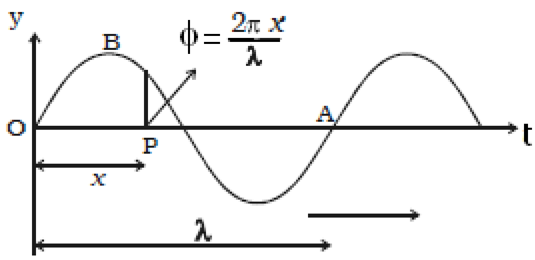
A progressive wave is defined as the onward transmission of the vibratory motion of a body in an elastic medium from one particle to the successive particle.

Equation of a plane progressive wave

An equation can be formed to represent generally the displacement of a vibrating particle in a medium through which a wave passes.

Thus each particle of a progressive wave executes simple harmonic motion of the same period and amplitude differing in phase from each other.

Let us assume that a progressive wave travels from the origin O along the positive direction of X axis, from left to right



The displacement of a particle at a given instant is y = A sinωt where *a* is the amplitude of the vibration of the particle and ω = 2π*f*.

The displacement of the particle P at a distance *x* from O at a given instant is given by,

y = *a* sin (ωt - ϕ).

If the two particles are separated by a distance λ, they will differ by a phase of 2π. Therefore, the phase ϕ of the particle P at a distance x is

$$\varphi = \frac{2\pi x}{\lambda}$$

$$y = A sin\,(\omega t - \frac{2\pi x}{\lambda})$$

$$But\ k = \frac{2\pi}{\lambda}\ wave\ vector$$

$$y = A sin(\omega t - kx)\ \ ---(1)$$

$$But\ \omega = \frac{2\pi}{T}$$

$$y = A\sin\left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda}\right)$$

$$y = A\sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) ----(2)$$

If the wave travels in opposite direction, the equation becomes

$$y = A\sin 2\pi\left(\frac{t}{T} + \frac{x}{\lambda}\right)$$

(i) Variation of phase with time

The phase changes continuously with time at a constant distance. At a given distance *x* from O let ϕ_1 and ϕ_2 be the phase of a particle at time t_1 and t_2 respectively.

$$\varphi_1 = 2\pi\left(\frac{t_1}{T} - \frac{x}{\lambda}\right)$$

$$\varphi_2 = 2\pi\left(\frac{t_2}{T} - \frac{x}{\lambda}\right)$$

$$\varphi_2 - \varphi_1 = \frac{2\pi}{T}(t_2 - t_1)$$

$$\Delta\varphi = \frac{2\pi}{T}\Delta t$$

This is the phase change $\Delta\phi$ of a particle in time interval Δt. If $\Delta t = T$, $\Delta\phi = 2\pi$. This shows that after a time period T, the phase of a particle becomes the same

Thus $d\varphi/dt$ = constant

$$\therefore \frac{d}{dt}(\omega t - kx) = 0$$

$$\omega - k\frac{dx}{dt} = 0$$

$$\frac{dx}{dt} = \frac{\omega}{k} = v$$

Here v is the phase speed of wave. Which is same as speed of wave

(ii) Variation of phase with distance

At a given time t phase changes periodically with distance *x*. Let ϕ_1 and ϕ_2 be the phase of two particles at distance x_1 and x_2 respectively from the origin at a time t.

$$\varphi_1 = 2\pi\left(\frac{t}{T} - \frac{x_1}{\lambda}\right)$$

$$\varphi_2 = 2\pi\left(\frac{t}{T} - \frac{x_2}{\lambda}\right)$$

$$\varphi_2 - \varphi_1 = -\frac{2\pi}{T}(x_2 - x_1)$$

$$\Delta\varphi = -\frac{2\pi}{T}\Delta x$$

The negative sign indicates that the forward points lag in phase when the wave travels from left to right. When $\Delta x = \lambda$, $\Delta\phi = 2\pi$, the phase difference between two particles having a path difference λ is 2π

Characteristics of progressive wave

1. Each particle of the medium executes vibration about its mean position. The disturbance progresses onward from one particle to another.
2. The particles of the medium vibrate with same amplitude about their mean positions.
3. Each successive particle of the medium performs a motion similar to that of its predecessor along the propagation of the wave, but later in time.
4. The phase of every particle changes from 0 to 2π.
5. No particle remains permanently at rest. Twice during each vibration, the particles are momentarily at rest at extreme positions, different particles attain the position at different time.
6. Transverse progressive waves are characterised by crests and troughs. Longitudinal waves are characterised by compressions and rarefactions.
7. There is a transfer of energy across the medium in the direction of propagation of progressive wave.
8. All the particles have the same maximum velocity when they pass through the mean position.
9. The displacement, velocity and acceleration of the particle separated by mλ are the same, where m is an integer.

Intensity and sound level

The loudness of a sound depends on intensity of sound wave and sensitivity of the ear. The intensity is defined as the amount of energy crossing per unit area per unit time perpendicular to the direction of propagation of the wave.
Intensity is measured in W m^{-2}.
The intensity of sound depends on (i) Amplitude of the source ($I \propto A^2$),
(ii) Surface area of the source ($I \propto S$),
(iii) Density of the medium ($I \propto \rho$),
(iv) Frequency of the source ($I \propto f^2$) and
(v) Distance of the observer from the source ($I \propto 1/r^2$

Solved Numerical

Q) A simple harmonic wave has the equation y = 0.3sin(314t – 1.57x), where t, x and y are in seconds, meteres and cm respectively. Find the frequency and the wavelength of the wave. Another wave has the equation y' = 0.10 sin (314t – 1.57x + 1.57). Deduce the phase difference and the ratio of intensities of wave

Solution:

Since y is in cm thus given equation in meters unit is

$$y = \frac{0.3}{100} sin(314t - 1.57x)$$

Comparing with standard equation y= a sin(ωt –kx)

We get

$$\omega = 2\pi f = 314$$

Frequency:

$$f = \frac{314}{2 \times 3.14} = 50Hz$$

Wavelength

$$k = \frac{2\pi}{\lambda}$$

$$\lambda = \frac{2\pi}{k} = \frac{2 \times 3.14}{1.57} = 4m$$

Phase difference between two wave is (314t – 1.57x + 1.57) – (314t – 1.57x) = 1.57 radian

Or

$$\frac{1.57 \times 180}{\pi} = 90^o$$

The ratio of intensity

$$\frac{I_1}{I_2} = \frac{a_1^2}{a_2^2} = \frac{9}{1}$$

Q) Given the equation for a wave in a string y = 0.03sin(3x – 2t)

Where y and x are in metres and t is in seconds

(a) When t = 0, what is the displacement at x = 0.1 m?
(b) When x = 0.1m, what is the displacement at t=0 and t = 0.2s?
(c) What is the equation for the velocity of oscillation of particle of string and what is the maximum velocity
(d) What is the velocity of propagation of waves.

Solution:

Given equation y = 0.03sin(3x – 2t)

(a) At t=0 , x = 01m , y = 0.03sin(0.3) = 8.86×10^{-3}m

(b) X = 0.1 m and t = 0 y = 8.86×10^{-3}m

At x= 0.1 at t = 0.2s

Y = 0.03sin(0.3-0.4) = -0.03sin(0.1) m = -2.997×10^{-3}m

(c) Particle velocity

$$V_P = \frac{dy}{dt} = -0.06\cos(3x - 2t)$$

$|V_P|_{max} = 6\times10^{-2}$ m/s

(d) Wave velocity = ω/k = 2/3 = 0.67 m/s

Superposition principle

When two waves travel in a medium simultaneously in such a way that each wave represents its separate motion, then the resultant displacement at any point at any time is equal to the vector sum of the individual displacements of the waves.

Let us consider two simple harmonic waves of same frequency travelling in the same direction. If a_1 and a_2 are the amplitudes of the waves and ϕ is the phase difference between them, then their instantaneous displacements are

$$y_1 = Asin\omega t$$

$$y_1 = Asin(\omega t + \varphi)$$

According to the principle of superposition, the resultant displacement is represented by

$y = y_1 + y_2$

$= A_1 \sin \omega t + A_2 \sin (\omega t + \phi)$

$= A_1 \sin \omega t + A_2 (\sin \omega t. \cos \phi + \cos \omega t.\sin \phi)$

$= (A_1 + A_2 \cos \phi) \sin \omega t + a2 \sin \phi \cos \omega t$ ...(3)

Put $A_1 + A_2 \cos \phi = A \cos \theta$ ...(4)

$A_2 \sin \phi = A \sin \theta$ ...(5)

Where A and θ are constants, then

$y = A \sin \omega t. \cos \theta + A \cos \omega t. \sin \theta$

or $y = A \sin (\omega t + \theta)$ ...(6)

This equation gives the resultant displacement with amplitude A.

From eqn. (4) and (5)

$A^2 cos^2 \theta + A^2 sin^2 \theta = (A_1 + A_2 cos \phi)^2 + (A_2 sin \phi)^2$

$\therefore A^2 = A_1^2 + A_2^2 + 2A_1A_2 cos \phi$

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2cos\varphi}$$

And

$$tan\theta = \frac{A_2sin\varphi}{A_1 + A_2cos\varphi}$$

We know that intensity is directly proportional to the square of the amplitude

$I \propto A^2$

$\therefore I \, \alpha \, (A_1^2 + A_2^2 + 2A_1A2 \cos \phi)$

Special cases

The resultant amplitude A is maximum, when *cos* $\phi = 1$ *or* $\phi = 2m\pi$ where *m* is an integer
(i.e) $I_{max} \alpha (A_1 + A_2)^2$
The resultant amplitude A is minimum when
$\cos\phi = -1$ or $\phi = (2m + 1)\pi$
$I_{min} \alpha (A_1 - A_2)^2$
The points at which interfering waves meet in the same phase
$\phi = 2m\pi$ i.e $0, 2\pi, 4\pi, ...$ are points of maximum intensity, where constructive interference takes place.

Reflection of wave

Reflection of wave from a rigid support

Suppose a wave propagates in the decreasing value of x, represented by equation
$y = A \sin(\omega t + kx)$ reaches a point $x = 0$
When the wave arrives at rigid support, support exerts equal and opposite force on the sting. This reaction force generates a wave at the support which travels back (along increasing values of x) along the string . This wave is known as reflected wave
At the support x= 0. Resultant displacement due to incident and reflected wave is zero
Thus if incident wave is
$y_i = A\sin(\omega t)$ since $x = 0$
then according to super position principle reflected wave equation at x= 0 is
$y_r = -A\sin(\omega t)$
or $y_r = A\sin(\omega t + \pi)$
Above equation shows that during reflection of wave phase increase by π
The reflected wave is travelling in positive x direction so the equation for reflected wave may be written as
$y_r = A \sin(\omega t + \pi - kx)$
$y_r = -A \sin(\omega t - kx)$

If incident wave equation is $y_i = A\sin(\omega t - kx)$ the equation for reflected wave is
$y_r = -A\sin(\omega t + kx)$
(b)Reflection of waves from a free end:
Suppose one end of the string is tied to a very light ring which can move or slide on the vertical rod without friction. Such end is called free end
Suppose crest reaches such free end then ring moves in upward direct as it is not fixed. As a result phase of reflected wave is same of incident wave. Or phase of reflected and incident wave is same.
If $y_i = A\sin(\omega t + kx)$ represent incident wave then
$Y_r = - A\sin(\omega t - kx)$ represent reflected wave

Beats

When two waves of nearly equal frequencies travelling in a medium along the same direction superimpose upon each other, beats are produced. The amptitude of the resultant sound at a point rises and falls regularly.
The intensity of the resultant sound at a point rises and falls regularly with time. When the intensity rises to maximum we call it as waxing of sound, when it falls to minimum we call it as waning of sound.
The phenomenon of waxing and waning of sound due to interference of two sound waves of nearly equal frequencies are called beats.
The number of beats produced per second is called beat frequency, which is equal to the difference in frequencies of two waves.

Analytical method
Let us consider two waves of slightly different frequencies f_1 and f_2 ($f_1 > f_2$) having equal amplitude travelling in a medium in the same direction. At time t = 0, both waves travel in same phase. The equations of the two waves are
$y_1 = a \sin 2\pi f_1 t$ and $y_1 = a \sin 2\pi f_2 t$
When the two waves superimpose, the resultant displacement is given by
$y = y_1 + y_2$
$y = a \sin (2\pi f_1) t + a \sin (2\pi f_2) t$

$$y = 2a \sin 2\pi \left(\frac{f_1 + f_2}{2}\right) t \cos 2\pi \left(\frac{f_1 - f_2}{2}\right) t$$

Substituting

$$A = 2a \cos 2\pi \left(\frac{f_1 - f_2}{2}\right) \quad and\ f = \frac{f_1 + f_2}{2}$$

$$y = A \sin 2\pi f t$$

This represents a simple harmonic wave of frequency $f = \frac{f_1 + f_2}{2}$
and amplitude A which changes with time.

(i) The resultant amplitude is maximum (i.e) $\pm 2a$, if

$$\cos 2\pi \left(\frac{f_1 - f_2}{2}\right) = \pm 1$$

$$2\pi \left(\frac{f_1 - f_2}{2}\right) = n\pi$$

(where n = 0, 1, 2 ...) or $(f_1 - f_2)\, t = n$
The first maximum is obtained at $t_1 = 0$
The second maximum is obtained at

$$t_2 = \frac{1}{f_1 - f_2}$$

The third maximum at

$$t_3 = \frac{2}{f_1 - f_2}$$

and so on.

The time interval between two successive maxima is

$t_2 - t_1 = t_3 - t_2 = 1 / (f_1 - f_2)$

Hence the number of beats produced per second is equal to the reciprocal of the time interval between two successive maxima.

(ii) The resultant amplitude is minimum (i.e) equal to zero, if

$$cos 2\pi \left(\frac{f_1 - f_2}{2}\right) = 0$$

$$2\pi \left(\frac{f_1 - f_2}{2}\right) = \frac{\pi}{2} + n\pi$$

$$(f_1 - f_2)t = \frac{2n + 1}{2}$$

where n = 0, 1, 2 ...

The first minimum is obtained at n = 0

$$t'_1 = \frac{1}{2(f_1 - f_2)}$$

The second minimum is obtained at n = 1

$$t'_2 = \frac{2}{2(f_1 - f_2)}$$

The third minimum is obtained at

$$t'_3 = \frac{5}{2(f_1 - f_2)}$$

Time interval between two successive minima is

$$t'_2 - t'_1 = t'_3 - t'_2 = \frac{1}{f_1 - f_2}$$

Hence, the number of beats produced per second is equal to the reciprocal of time interval between two successive minima.

Uses of beats

(i) The phenomenon of beats is useful in tuning two vibrating bodies in unison. For example, a sonometer wire can be tuned in unison with a tuning fork by observing the beats. When an excited tuning fork is kept on the sonometer and if the sonometer wire is also excited, beats are heard, when the frequencies are nearly equal. If the length of the wire is adjusted carefully so that the number of beats gradually decreases to zero, then the two are said to be in unison. Most of the musical instruments are made to be in unison based on this method.

(ii) The frequency of a tuning fork can be found using beats. A standard tuning fork of frequency N is excited along with the experimental fork. If the number of beats per second is n, then the frequency of experimental tuning fork is N+n. The experimental tuning fork is then loaded with a little bees' wax, thereby decreasing its frequency.
Now the observations are repeated. If the number of beats increases, then the frequency of the experimental tuning fork is N-n, and if the number of beats decreases its frequency is N + n.

Solved Numerical

Q) Two wires are fixed on a sonometer. Their tensions are in ratio 8:1, the lengths in the ratio 36:35, the diameters in the ratio 4:1, the densities in the ratio 1:2. Find the frequency of beats if the note of higher pitch has a frequency of 360Hz.

Solution:

$$v = \sqrt{\frac{T}{m}}$$

T is tension and m is mass of string per unit length

$$v = \sqrt{\frac{T}{\left(\frac{\pi d^2}{4}\right)\rho}}$$

$$\frac{v_1}{v_2} = \sqrt{\frac{T_1}{T_2}\frac{d_2^2}{d_1^2}\frac{\rho_2}{\rho_1}} = \sqrt{\left(\frac{8}{1}\right)\left(\frac{1}{16}\right)\left(\frac{2}{1}\right)} = 1$$

Thus $v_1 = v_2 = v$(say)

$$f_1 = \frac{v}{2L_1} = 360 - x$$

$$f_2 = \frac{v}{2L_2} = 360$$

$$\frac{360 - x}{360} = \frac{L_2}{L_1} = \frac{35}{36}$$

360 – x = 350

X = 10 beats

Q) A wire of a sonometer 1m long weight 5g and is stretched by a force of 10kg wt. When the length of the vibrating portion of wire is 28 cm three beats per second are heard, if the wire and unkown frequency are sound together. The wire is slightly shorted and 4 beats per second are then heard. What is the frequency of the fork?

Solution:

The frequency of the sonometer wire

$$f_1 = \frac{1}{2L}\sqrt{\frac{T}{m}}$$

L = 0.28m, T = 10×9.8 = 98 N, m = 5gm = 5×10^{-3} kg/m

$$f_1 = \frac{1}{2 \times 0.28}\sqrt{\frac{98}{5 \times 10^{-3}}} = 250Hz$$

The sonometer wire has a frequency which differs from the tuning fork by 3Hz.
If the wire is shorted, the frequency of the wire increases. The number of beats also increases to 4.
His means that the frequency of the wire is greater than the frequency of the tuning fork
∴ the frequency of the tuning fork = 250 -3 =247 Hz

Stationary waves

When two progressive waves of same amplitude and wavelength travelling along a straight line in opposite directions superimpose on each other, stationary waves are formed.

Analytical method

Let us consider a progressive wave of amplitude *a* and wavelength λ travelling in the direction of X axis.

$$y_1 = a sin2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right)$$

This wave is reflected from a ***free end*** and it travels in the negative direction of X axis, then

$$y_1 = a sin2\pi \left(\frac{t}{T} + \frac{x}{\lambda}\right)$$

According to principle of superposition, the resultant displacement is $y = y_1 + y_2$

$$y = a \left[sin2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right) + sin2\pi \left(\frac{t}{T} + \frac{x}{\lambda}\right)\right]$$

$$y = a \left[2sin \frac{2\pi t}{T} cos \frac{2\pi x}{\lambda}\right]$$

$$y = 2a cos \frac{2\pi x}{\lambda} sin \frac{2\pi t}{T}$$

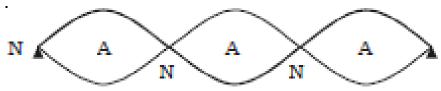
This is the equation of a stationary wave.
(i) At points where *x* = 0, λ/2, λ, 3λ/2 the values of

$$cos \frac{2\pi x}{\lambda} = \pm 1$$

∴ A = + $2a$. At these points the resultant amplitude is maximum. They are called *antinodes* (Fig.).
(ii) At points where *x* = λ/4, 3λ/4 , 5λ/4 the values of

$$\cos \frac{2\pi x}{\lambda} = 0$$

$\therefore A = 0$. At these points the resultant amplitude is minimum. They are called *nodes* (Fig.).



The distance between any two successive antinodes or nodes is equal to λ/2 and the distance between an antinode and a node is λ/4

(iii) When t = 0, T/2, T, 3T/2, …

$$\sin \frac{2\pi t}{T} = 0$$

Displacement is zero

(iv) When t = T/4, 3T/4, 5T/4, …..

$$\sin \frac{2\pi t}{T} = \pm 1$$

Displacement is maximum

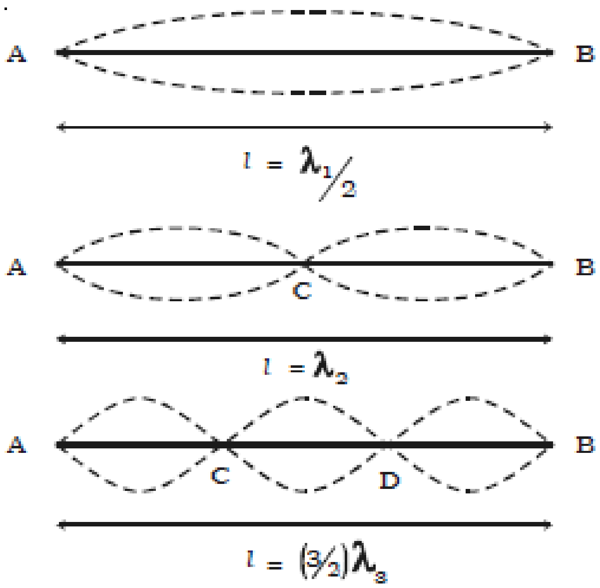
Characteristics of stationary waves

1. The waveform remains stationary.
2. Nodes and antinodes are formed alternately.
3. The points where displacement is zero are called nodes and the points where the displacement is maximum are called antinodes.
4. Pressure changes are maximum at nodes and minimum at antinodes.
5. All the particles except those at the nodes, execute simple harmonic motions of same period.
6. Amplitude of each particle is not the same, it is maximum at antinodes decreases gradually and is zero at the nodes.
7. The velocity of the particles at the nodes is zero. It increases gradually and is maximum at the antinodes.
8. Distance between any two consecutive nodes or antinodes is equal to λ / 2 whereas the distance between a node and its adjacent antinode is equal to λ/4 .
9. There is no transfer of energy. All the particles of the medium pass through their mean position simultaneously twice during each vibration
10. Particles in the same segment vibrate in the same phase and between the neighboring segments, the particles vibrate in opposite phase.

Modes of vibration of stretched string

(i) Fundamental frequency

If a wire is stretched between two points, a transverse wave travels along the wire and is reflected at the fixed end. A transverse stationary wave is thus formed as shown in Fig. When a wire AB of length L is made to vibrate in one segment then



$$L = \frac{\lambda_1}{2}$$

$\lambda_1 = 2L$. This gives the lowest frequency called fundamental frequency

$$f_1 = \frac{v}{\lambda_1}$$

We know that velocity of wave in stretched sting is given by $v = \sqrt{\frac{T}{m}}$

$$f_1 = \frac{1}{2L}\sqrt{\frac{T}{m}}$$

(ii) Overtones in stretched string

If the wire AB is made to vibrate in two segments then $L = \lambda_2$

But

$f_2 = \frac{v}{\lambda_2}$

$$f_2 = \frac{1}{L}\sqrt{\frac{T}{m}}$$

f_2 is the frequency of the first overtone.

Since the frequency is equal to twice the fundamental, it is also known as second harmonic.

Similarly, higher overtones are produced, if the wire vibrates with more segments.

If there are *n* segments, the length of each segment is

$$\lambda_n = \frac{2L}{n}$$

Frequency f_n

$$f_n = \frac{n}{2L}\sqrt{\frac{T}{m}} = nf_1$$

(i.e) n^{th} harmonic corresponds to $(n-1)^{th}$ overtone

Laws of transverse vibrations of stretched strings

The laws of transverse vibrations of stretched strings are

(i) the law of length

For a given wire (*m* is constant), when *T* is constant, the fundamental frequency of vibration is inversely proportional to the vibrating length (i.e)

$$f \propto \frac{1}{L} \quad or\ fL = constant$$

(ii) law of tension

For constant *L* and *m,* the fundamental frequency is directly proportional to the square root of the tension (i.e) $n \propto \sqrt{T}$.

(iii) the law of mass.

For constant *L* and *T*, the fundamental frequency varies inversely as the square root of the mass per unit length of the wire (i.e) $n \propto 1/\sqrt{m}$

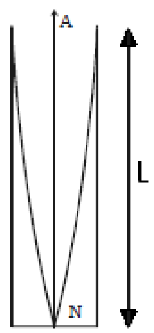
Vibrations of air column in pipes

Musical wind instruments like flute, clarinet etc. are based on the principle of vibrations of air columns. Due to the superposition of the incident wave and the reflected wave, longitudinal stationary waves are formed in the pipe.

Organ pipes

Organ pipes are musical instruments which are used to produce musical sound by blowing air into the pipe. Organ pipes are two types (i) closed organ pipes, closed at one end (ii) open organ pipe, open at both ends.

(i) Closed organ pipe : If the air is blown lightly at the open end of the closed organ pipe, then the air column vibrates (Fig.a) in the fundamental mode. There is a node at the closed end and an antinode at the open end. If *L* is the length of the tube, $\lambda_1 = 4L$



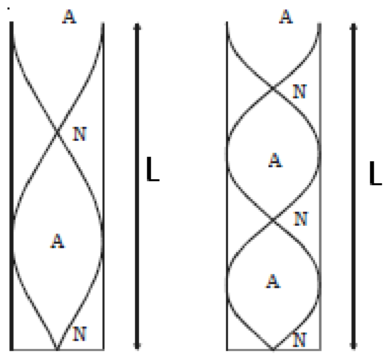
$\lambda_1 = 4L$
(a)

If f_1 is the fundamental frequency of the vibrations and *v* is the velocity of sound in air, then

$$f_1 = \frac{v}{\lambda_1} = \frac{v}{4L}$$

If air is blown strongly at the open end, frequencies higher than fundamental frequency can be produced. They are called overtones.

b & c shows the mode of vibration with two or more nodes and antinodes



$\lambda_3 = 4L/3$ (b) $\lambda_5 = 4L/5$ (c)

L =3λ$_3$/4 or λ$_3$ = 4L/3

$$\therefore f_3 = \frac{v}{\lambda_3} = \frac{3v}{4L} = 3f_1$$

This is the first overtone or third harmonics
Similarly

$$f_5 = \frac{5v}{4L} = 5f_1$$

This is the second overtone or fifth harmonic

Solved Numerical

Q) A disc contains 30 small holes evenly distributed along the rim and is rotated at the uniform rate of 540 revolutions per minute. A jet of air is blown through the hole in to a pipe whose other end is closed by movable piston. The length of the pipe can be varied between 90cm to 120 cm. What should be the exact length of the pipe so that the sound produced at the frequency on interruption of air jet is loudest? (velocity of sound in air =330 m/s)

Solution:

The frequency of notes produced by the disc = number of holes × frequency of rotating disc

$$f = 60 \times \frac{540}{60} = 270\ Hz$$

This note is produced at the mouth of the closed pipe
Let us first assume that closed pipe is vibrates in its fundamental mode
Then fundamental frequency

$$f_1 = \frac{v}{4L}$$

$$L = \frac{v}{4f_1}$$

$$L = \frac{330}{4 \times 270} = 0.306m = 30.6\ cm$$

This length is much shorter than the length given
The same fundamental frequency can also occur at a length
From the formula for first over tone

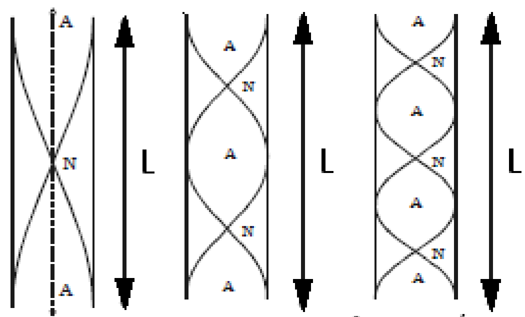
$$f_3 = \frac{3v}{4L}$$

if length is changed from L to 3L then we get

$$f = \frac{3v}{4(3L)} = \frac{v}{4L} = f_1$$

Thus the same fundamental frequency can also occur at a length 3 ×30.6 = 91.8 cm. This is greater than the minimum length 90cm of tube. The tube should be adjusted to this length to get the loudest note as desired
Note: If in place of air if any other gas is used then calculate velocity of sound in that gas and used in place of v to determine length

(ii) Open organ pipe –



$\lambda_1 = 2L$ (a) $\lambda_2 = L$ (b) $\lambda_3 = 2L/3$ (c)

When air is blown into the open organ pipe, the air column vibrates in the fundamental mode Fig.a. Antinodes are formed at the ends and a node is formed in the middle of the pipe. If L is the length of the pipe, then λ_1=2L
$v = f_1\lambda_1 = f_1 2L$
The fundamental frequency

$$f_1 = \frac{v}{2L}$$

In the next mode of vibration additional nodes and antinodes are formed as shown in fig(b) and fig(c)
$L = \lambda_2$ or $v = f_2\lambda_2 = f_2L$

$$f_2 = \frac{v}{L} = 2f_1$$

Similarly,

$$f_2 = \frac{v}{\lambda_3} = \frac{3v}{2L} = 3f_1$$

This is the second overtone or third harmonic.

Therefore the frequency of nth overtone is *(n + 1) f_1* where *f_1* is the fundamental frequency. The frequencies of harmonics are in the ratio of 1 : 2 : 3 ....

Solved Numerical

Q) An open pipe filled with air has a fundamental frequency of 500Hz. The first harmonic of another organ pipe closed at one end and filled with carbon dioxide has the same frequency as that of the first harmonic of the open organ pipe. Calculate the length of each pipe. Assume that the velocity of sound in air and in carbondioxide to be 330 m/s and 264 m/s respectively

Solution:

The fundamental frequency is the first harmonics. In case of open pipe containing air at 30^{O}C. Let L_O be the length of the pipe. Then

$$f_1 = \frac{v}{2L_O}$$

$$L_O = \frac{v}{2f}$$

$$L_O = \frac{330}{2 \times 500} = 33\ cm$$

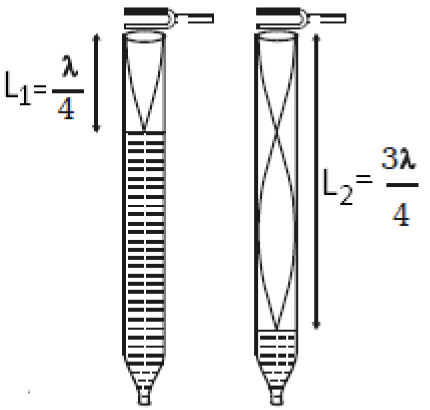
Let L_C be the length of the closed pipe. For the fundamental frequency of the ipipe

$$f_1 = \frac{v_{CO_2}}{4L_C}$$

$$L_C = \frac{264}{4 \times 500} = 13.2\ cm$$

Resonance air column apparatus

The glass tube is mounted on a vertical stand with a scale attached to it. The glass tube is partly filled with water. The level of water in the tube can be adjusted by raising or lowering the reservoir.



A vibrating tuning fork of frequency f is held near the open end of the tube.

The length of the air column is adjusted by changing the water level. The air column of the tube acts like a closed organ pipe.

When this air column resonates with the frequency of the fork the intensity of sound is maximum.

Here longitudinal stationary wave is formed with node at the water surface and an antinode near the open end. If L_1 is the length of the resonating air column

$$\frac{\lambda}{4} = L_1 + e \quad ---eq(1)$$

where e is the end correction.

The length of air column is increased until it resonates again with the tuning fork. If L_2 is the length of the air column

$$\frac{3\lambda}{4} = L_2 + e - - - eq(2)$$

From equations (1) and (2)

$$\frac{\lambda}{2} = L_2 - L_1$$

The velocity of sound in air at room temperature

$$v = f\lambda = 2f\,(L_2 - L_1)$$

End correction

The antinode is not exactly formed at the open end, but at a small distance above the open end. This is called the end correction

It is found that e = 0.61r*,* where *r* is the radius of the glass tube.

Doppler Effect

When a sound source and an observer are in relative motion with respect to the medium in which the wave propagate, the frequency of wave observed is different from the frequency of sound emitted by the source. This phenomenon is called Doppler effect. This is due to the wave nature of sound propagation and is therefore applicable to light waves also.

Calculation of apparent frequency

Suppose V is the velocity of sound in air, V_o is the velocity of observer (O) and f is the frequency of the source

(i) Source moves towards stationary observer

If the source S were stationary, the f waves sent out in one second towards the observer O would occupy a distance V and the wave length would bt v/f

If S is moving with velocity V_s towards stationary observer, the f waves emitted in one second occupy a distance (V-V_s) because S has moves a distance V_s towards O in 1 sec..

So apparent frequency would be

$$\lambda' = \left(\frac{V - V_s}{f}\right)$$

∴ apparent frequency

$$f' = \frac{velocity\ of\ sound\ relative\ to\ O}{wavelength\ of\ wave\ reaching\ O}$$

$$f' = \frac{V}{\lambda'} = f\left(\frac{V}{V - V_s}\right)$$

(ii) Source moves away from stationary observer:

Apparent wave length

$$\lambda' = \left(\frac{V + V_s}{f}\right)$$

$$f' = \frac{V}{\lambda'} = f\left(\frac{V}{V + V_s}\right)$$

(iii) Observer moving towards stationary source

$$f' = \frac{velocity\ of\ sound\ relative\ to\ O}{wavelength\ of\ wave\ reaching\ O}$$

Velocity of sound relative to O = V +V_o

And wavelength of waves reaching O = V/f

$$f' = \frac{V + V_0}{V/f} = f\left(\frac{V + V_0}{V}\right)$$

(iv) Observer moves away from the stationary source:

Velocity of sound relative to O = V –V_o

And wavelength of waves reaching O = V/f

$$f' = \frac{V - V_0}{V/f} = f\left(\frac{V - V_0}{V}\right)$$

(v) Source and observer both moves towards each other

Velocity of sound relative to O = V +V_o

And wavelength of waves reaching O = (V-V_s)/f

$$f' = \frac{V + V_0}{\frac{V - V_s}{f}} = f\left(\frac{V + V_0}{V - V_s}\right)$$

(vi) Source and observer both are moving away from each other

Velocity of sound relative to O = V - V_o

And wavelength of waves reaching O = (V+V_s)/f

$$f' = \frac{V - V_0}{\frac{V + V_s}{f}} = f\left(\frac{V - V_0}{V + V_s}\right)$$

(vii) Source moves towards observer but observer moves away from source

Velocity of sound relative to O = V - V_o

And wavelength of waves reaching O = (V+V_s)/f

$$f' = \frac{V - V_0}{\frac{V + V_s}{f}} = f\left(\frac{V - V_0}{V + V_s}\right)$$

(viii) Source moves away from observer but observer moves towards source

Velocity of sound relative to O = V + V_o

And wavelength of waves reaching O = (V+V_s)/f

$$f' = \frac{V + V_0}{\frac{V + V_s}{f}} = f\,(\frac{V + V_0}{V + V_s})$$

General equation can be written as

$$\boldsymbol{f' = f\,(\frac{V \pm V_o}{V \pm V_s})}$$

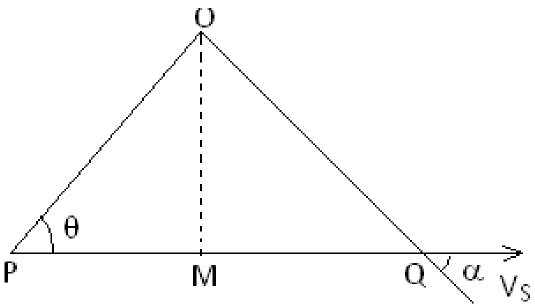
If **observer** moving **towards** source V_o is +Ve
If **observer** I moving **away** from source V_0 is –Ve

If **source** is moving **towards** or approaching observer V_s is -Ve
If **source** moving **away** from observer V_s is +Ve

Effect of wind velocity

If wind velocity(w) is in the direction of sound(v) then we can add wind velocity
If wind velocity (w) is opposite in the direction of sound we can subtract wind velocity for final velocity of sound

Doppler effect when the source is moving at an angle to the observer



Let O be a stationary observer and let a source of sound of frequency f be moving along the line PQ with constant speed $_s$
When the source is at O, the line PO makes angle θ with PQ, which is the direction of V_s
The component of velocity V_s along PO is $V_s\cos\theta$ and it is towards the observer

The apparent frequency in this case

$$f_a = f\,(\frac{V}{V - V_s\cos\theta})$$

As the source moves along PQ, θ increases cosθ decreases and the apparent frequency continuously diminishes. At M, θ=90^{O} and hence f_a = f
When the source is at Q, the component of velocity V_s is $V_s\cos\alpha$ which is directed away from the observer. Hence the apparent frequency

$$f_a = f\,(\frac{V}{V + V_s\cos\alpha})$$

Solved Numerical

Q) A train travelling at a speed of 20 m/s and blowing a whistle with frequency of 240 Hz is approaching a train B which is at rest. Assuming the speed of sound to be 340 m/s calculate the following
(a) Wavelength in air(i) in front and (ii) behind the train A

(b) Frequencies measured by a listener in train B while train A is (i)approaching and (ii) receding from train B
(c)If train B starts moving with speed of 10 m/s, what will be the frequencies heard bya passenger in train B if both were (i)approaching and (ii) receding
Solution:
(a)(i) The train A is approaching the train B which is at rest
Thus $V_O = 0$, since source is approaching thus V_S is -Ve

$$f' = f\left(\frac{V \pm V_o}{V \pm V_s}\right)$$

$$f' = f\left(\frac{V}{V - V_s}\right)$$

Now f = V/λ

$$\therefore \frac{V}{\lambda'} = \frac{V}{\lambda}\left(\frac{V}{V - V_s}\right)$$

$$\therefore \lambda' = \lambda\left(\frac{V - V_s}{V}\right) = \frac{\lambda}{V}(V - V_s)$$

$$\therefore \lambda' = \frac{1}{f}(V - V_s)$$

$$\therefore \lambda' = \frac{1}{240}(340 - 20) = 1.33\ m$$

(ii)For observer behind the train A since source is moving away V_S is positive Thus

$$\therefore \lambda' = \frac{1}{f}(V + V_s)$$

$$\therefore \lambda' = \frac{1}{240}(340 + 20) = 1.5\ m$$

(b)(i)Frequency as measured by listener un train B when A is approaching B
Source is approaching V_S is –Ve, Listener is stationary $V_O = 0$

$$f' = f\left(\frac{V \pm V_o}{V \pm V_s}\right)$$

$$\therefore f' = f\left(\frac{V}{V - V_s}\right)$$

$$\therefore f' = 240\left(\frac{340}{340 - 20}\right) = 255\ Hz$$

(ii) Frequency as measured by listener in B as A recedes from hin
Source moving away V_S is +Ve, Listener is stationary $V_O = 0$

$$f' = f\left(\frac{V \pm V_o}{V \pm V_s}\right)$$

$$\therefore f' = f\left(\frac{V}{V+V_s}\right)$$

$$\therefore f' = 240\left(\frac{340}{340+20}\right) = 227\ Hz$$

(C)(i) Now both the trains are approaching each other
observer moving towards source V_o is +Ve and
Source is approaching V_S is –Ve

$$f' = f\left(\frac{V+V_o}{V-V_s}\right)$$

$$\therefore f' = f\left(\frac{V+V_o}{V-V_s}\right)$$

$$\therefore f' = 240\left(\frac{340+10}{340-20}\right) = 263\ Hz$$

(ii) Now both trains are moving away
observer moving away source V_o is -Ve and
Source is moving away V_S is +Ve

$$f' = f\left(\frac{V+V_o}{V-V_s}\right)$$

$$\therefore f' = f\left(\frac{V-V_o}{V+V_s}\right)$$

$$\therefore f' = 240\left(\frac{340-10}{340+20}\right) = 220\ Hz$$

Q) A spectroscope examination of light from a certain star shows that the apparent wavelength of certain spectral line from a certain star is 5001Å. Whereas the observed wavelength of the same line produced by terrestrial source is 5000. In what direction and what speed do these figure suggest that the star is moving relative to the earth.

<u>Solution:</u>

Actual wavelength of the spectral line = 5000 Å
Apparent wave length of the same line = 5001 Å
Since wavelength increases, the star is moving away from earth (red shif) V_s is positive
Observer is stationary and velocity of light V = c

$$f' = f\left(\frac{V \pm V_o}{V \pm V_s}\right)$$

$$f' = f\left(\frac{c}{c+V_s}\right)$$

$$\frac{c}{\lambda'} = \frac{c}{\lambda}\left(\frac{c}{c+V_s}\right)$$

$$\lambda' = \lambda\left(\frac{c+V_s}{c}\right)$$

$$V_S = \frac{\lambda' c}{\lambda} - c$$

$$V_S = c\left(\frac{\lambda' - \lambda}{\lambda}\right)$$

$$V_S = 3 \times 10^{8}\left(\frac{10^{-10}}{5000 \times 10^{-10}}\right) = 6 \times 10^{4}\ m/s$$

Q) A whistle of frequency 1000 Hz is blown continuously in front of a board made of plaster of paries. If the board is made to move away from the whistle with a velocity of 1.375 m/s, calculate the number of beats heard per second by a stationary observer situated infront of the board in line with the whistle (velocity of sound in air = 330 m/s)

Solution

The frequency of whistle = 1000Hz

The reflecting board is moving away from the whistle with velocity of 1.375 m/s

The reflected source of sound for the observer is the image of the whistle behind the board, the image is moving away from the board with velocity 2×1.375 = 2.750 m/s

Therefore the frequency of sound heard by the observer due to reflection from the board is

$$f' = f\left[\frac{V}{V+V_S}\right]$$

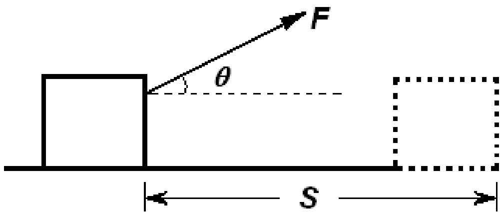
$$f' = 1000\left[\frac{330}{330+2.75}\right] = 992\ (approx)$$

∴ number of beats heard by the observer = 1000 - 992 = 8 per second

WORK, ENERGY AND POWER

Work done by a constant force:

Consider an object undergoes a displacement S along a straight line while acted on a force F that makes an angle θ with S as shown



The work done W by the agent is the product of the component of force in the direction of displacement and the magnitude of displacement

W = FScosθ

Work done is a scalar quantity and S.I. unit is N-m or Joule (J). Its dimensional formula is $M^1L^2T^{-2}$

We can also write; work done as a scalar product of force and displacement

$$W = \mathbf{F} \cdot \mathbf{S}$$

From this definition, we conclude the following points:

(i) Work done by a force is zero, if point of application of force does not move (S=0)

(ii) Work done by a force is zero if displacement is perpendicular to the force ($\theta=90^O$)

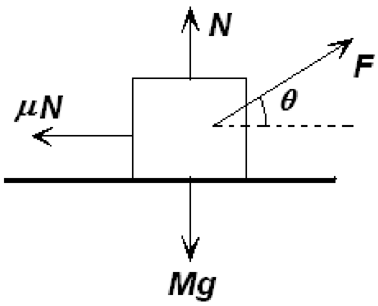
(iii) If angle between force and displacement is acute ($\theta < 90^O$), we say that work done by the force is positive or work is done on the object

(iv) If angle between force and displacement is obtuse ($\theta > 90^O$), we say that work done by the force is negative or work is done by the object

Solved Numerical

Q) A block of mass M is pulled along a horizontal surface by applying a force at an angle θ with horizontal. Coefficient of friction between block and surface is μ. If the block travels with uniform velocity, find the work done by this applied force during a displacement d of the block

Solution



The forces acting on the block as shown in figure Force F will resolve as Fsinθ along normal while Fcosθ will be opposite to friction. Thus we get

Fcosθ = μN (1)

And

N + Fsinθ = Mg--------- eq(2)

Eliminating N from equation (1) and (2)

Fcosθ = μ (Mg - Fsinθ)

$$F\cos\theta + F\sin\theta = \mu mg$$

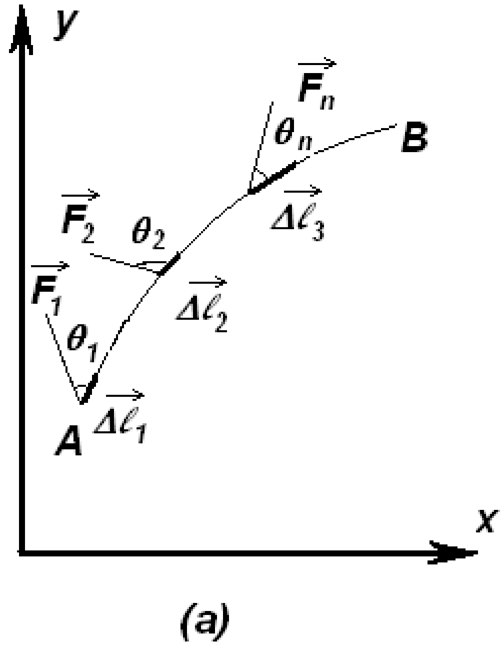
$$F = \frac{\mu Mg}{\cos\theta + \sin\theta}$$

Work done by this force during displacement d

$$W = Fd = \frac{\mu Mgd}{\cos\theta + \sin\theta}$$

WORK, ENERGY AND POWER

Work done by Variable force



(a)

Consider a particle being displaced along the curved path under the action of a varying force, as shown in figure. In such situation, we cannot use W= (Fcosθ)S to calculate the work done by the force because this relationship applies when F is constant in magnitude and direction

However if we imagine that the particle undergoes a very small displacement $\Delta \mathbf{l_1}$, shown in figure(a), then F is approximately constant over this interval and we can express the work done by the force for this small displacement as $W_1 = F_x \Delta \mathbf{l_1}$

In order to calculate work done, the whole curved path is assumed to be divided in small segments $\Delta \mathbf{l_1}$, $\Delta \mathbf{l_2}$, $\Delta \mathbf{l_3}$,--------- $\Delta \mathbf{l_n}$

Let $\mathbf{F_1}$, $\mathbf{F_2}$, $\mathbf{F_3}$, --------- $\mathbf{F_n}$ be the force at respective segments. The force over each such segment can be considered as constant because the segments are very small.

Total work done

$W = \mathbf{F_1}\cdot\Delta \mathbf{l_1} + \mathbf{F_2}\cdot\Delta \mathbf{l_2} + \mathbf{F_3}\cdot\Delta \mathbf{l_3} + \ldots\ldots\ldots\ldots + \mathbf{F_n}\cdot\Delta \mathbf{l_n}$

$$W = \sum_{A}^{B} \vec{F_i}\vec{\Delta l_i}$$

If we take $|\Delta \mathbf{l}| \rightarrow 0$, the above summation gets converted into an integral

$$W = \int_A^B F \cdot \vec{dl} = \int_A^B F\cos\theta dl$$

Solved Numerical

Q) A particle moves from x = 0 to x = 10m on X-axis under the effect of force

$F(x) = (3x^2 - 2x + 7)\mathbf{i}$ N.

Calculate the work done

Solution: since direction of force and displacement is same θ = 0

$$W = \int_0^{10} F\,dx$$

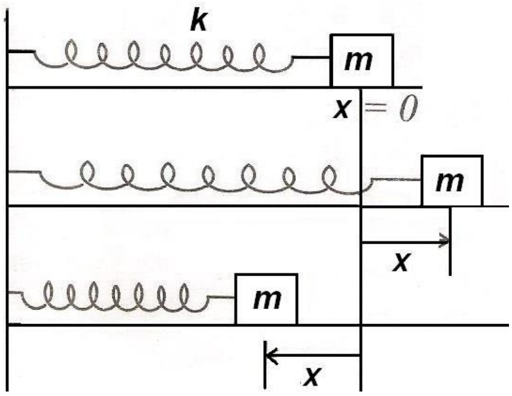
$$W = \int_0^{10} (3x^2 - 2x + 7)\,dx$$

$$W = \left[\frac{3x^3}{3}\right]_0^{10} - \left[\frac{2x^2}{2}\right]_0^{10} + [7x]_0^{10}$$

W = 1000 - 100 + 70 = 970 J

WORK, ENERGY AND POWER

Work done by a spring



A common physical system for which the force varies with position is a spring-block as shown in figure. If the spring is stretched or compressed by a small distance from its unstitched or compressed by a small distance from its unscratched configuration, the spring will exert a force on the block given by F = -kx, where x is compression or elongation in spring, k is a constant called spring constant whose value depends inversely on un-stretched length and the nature of material of spring.

Negative sign in above equation indicates that the direction of the spring force is opposite to x , the displacement of the free end.

Consider a spring block system as shown in figure and let us calculate work done by the spring when block is displaced by x_0

At any moment if elongation is x, then force on block by spring is kx towards left.

Therefore, work done by the spring when block further displaced by dx

dw = - kxdx (Negative sign indicates displacement is opposite to spring force)

Total work done by the spring

$$W = -\int_0^{x_0} kx\,dx = -\frac{1}{2}kx_0^2$$

Similarly, work done by the spring when it is given a compression x_0 is

$$-\frac{1}{2}kx_0^2$$

We can also say that work done by external agent

$$\frac{1}{2}kx_0^2$$

Power

If external force is applied to an point like object and if the work done by this force is ΔW in the time interval Δt, then the average power during this interval is defined as

$$P = \frac{\Delta W}{\Delta t}$$

The work done on the object contributes to increasing energy of the object. A more general definition of power is the time rate of energy transfer. This instantaneous power is the limiting value of the average power as Δt approaches zero

$$P = \frac{dW}{dt}$$

Where we have represented the infinitesimal value of the work done by dW (even though it is not a change and therefore not a differential). We know that

WORK, ENERGY AND POWER

$dW = F \cdot S$

Therefore the instantaneous power can be written as

$$P = \frac{dW}{dt} = F \cdot \frac{dS}{dt} = F \cdot v$$

The SI unit of power is Joule per second (J/s), also called watt (W)

1W = 1 J/s = $1 kgm^2s^{-3}$

Energy

A body is said to possess energy if it has the capacity to do work. When a body possessing energy does some work, part of its energy is used up.

Conversely if some work is done upon an object, the object will be given some energy.

Energy and work are mutually convertible.

Kinetic energy

Kinetic energy (K.E.) is the capacity of a body to do work by virtue of its motion

If a body of mass m has velocity v its kinetic energy is equivalent to the work, which an external force would have to do to bring the body from rest to its velocity v.

The numerical value of the kinetic energy can be calculated from the formula

$$K.E. = \frac{1}{2} mv^2$$

This formula can be derived as follows:

Consider a **constant force F** which acting on a mass m initially at rest, particle accelerate with constant velocity and attend velocity v after displacement of S .

For the formula

$v^2 – u^2 = 2as$

Initial velocity is zero

$v^2 = 2as$

Multiply both the sides by m

$mv^2 = 2mas$

$mv^2 = 2W$ [As work = FS = mas]

$W = (1/2)mv^2$

But Kinetic energy of body is equivalent to the work done in giving the velocity to the body

Hence K.E = $(1/2)mv^2$

Since both m and v^2 are always positive K.E is always positive and does not depend up on the direction of motion of body. Another equation for kinetic energy

$$E = \frac{1}{2} mv^2 = \frac{1}{2} \frac{m^2v^2}{m} = \frac{1}{2} \frac{p^2}{m}$$

Potential energy

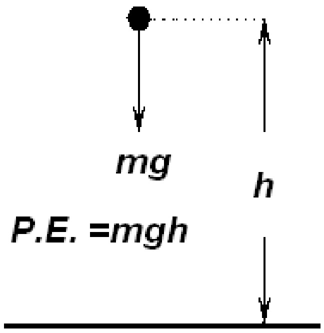
Potential energy is the energy due to position. If a body is in a position such that if it were released it would begin to move, it has potential energy

There are two common forms of potential energy, gravitational and elastic

Gravitational potential energy

It is possessed by virtue of height

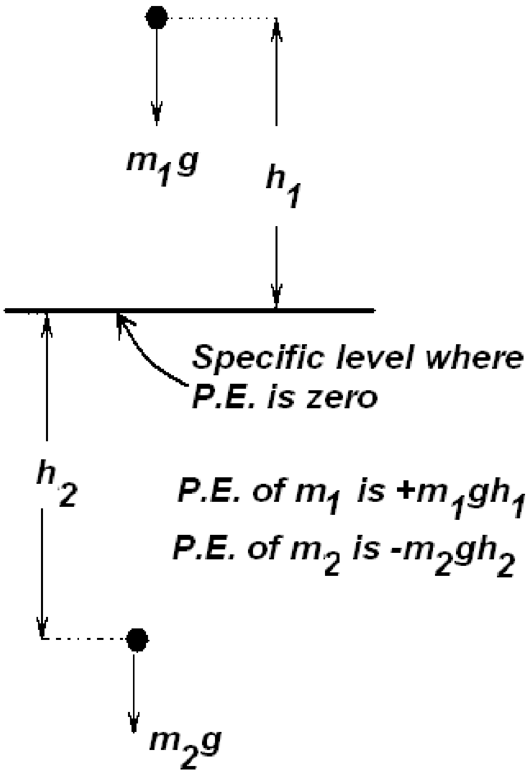
When an object is allowed to fall from one level to a lower level it gains speed due to gravitational pull, i.e. it gains kinetic energy. Therefore, in possessing height, a body has the ability to convert its gravitational potential energy into kinetic energy.



The gravitational potential energy is equivalent to the negative of the amount of work done by the weight of the body in causing the descent.

If a mass m is at a height h above a lower level, the P.E. possessed by the mass is (mg) (h)

Since h is the height of an object above a specific level, an object below the specified level has negative potential energy



Therefore Gravitational Potential Energy = $\pm$ mgh

- The chosen level from which height is measured has no absolute position. It is important therefore to indicate clearly the zero P.E. level in any problem in which P.E. is to be calculated.
- Gravitational Potential Energy = $\pm$ mgh is applicable only when h is very small in comparison to the radius of earth.

WORK, ENERGY AND POWER

Elastic potential Energy

It is a property of stretched or compressed springs.
The end of a stretched elastic spring will begin to move if it is released. The spring therefore possesses potential energy due to its elasticity (i.e. due to change in its configuration)
The amount of elastic potential energy stored in a spring of natural length a and spring constant k when it is extended by a length x is equal to the amount of work necessary to produce the extension
Work done = $(1/2)kx^2$ so
Elastic Potential energy = $(1/2)\ kx^2$
Elastic potential energy is never negative whether the spring is extended or compressed

Work energy theorem

When a body is acted upon by force acceleration is produced in it. Thus velocity of the body changes and hence the kinetic energy of the body also changes. Also force acting on a body displaces the body and so work is said to be done on the body by force. These facts indicate that there should be some relation between the work done on body and change in its kinetic energy.
The work done by the force F
W= F S
W = ma
W = mas
Also $v^2 - u^2 = 2as$
Multiplying both sides by m
$m(v^2 - u^2) = 2ams$

$$\frac{1}{2}mv^2 - \frac{1}{2}u^2 = mas$$

$$\frac{1}{2}mv^2 - \frac{1}{2}u^2 = W$$

Here u and v are the speeds before and after application of force.
The left hand side of above equation gives change in kinetic energy while right hand gives the work done
Thus ΔK = W
The work done by the resultant force on a body is equal to change in kinetic energy of the body. This statement is known as work energy theorem.

Work energy theorem for variable force
Work-energy theorem is valid from variable force
Suppose position dependent force F(x) acts on a body of mass m
Work done under the influence of force

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$$W = \int_i^f F(x)dx$$

$$W = \int_i^f m\frac{dv}{dt}dx$$

$$W = \int_i^f m\frac{dx}{dt}dv$$

$$W = \int_i^f mv\,dv \ [\because \ \frac{dx}{dt} = v]$$

$$W = m\int_i^f v\,dv$$

If initial velocity of the body and final velocity of the body are v_i and v_f

$$W = m\int_{v_i}^{v_f} v\,dv = m\,[\frac{v^2}{2}]_{v_i}^{v_f}$$

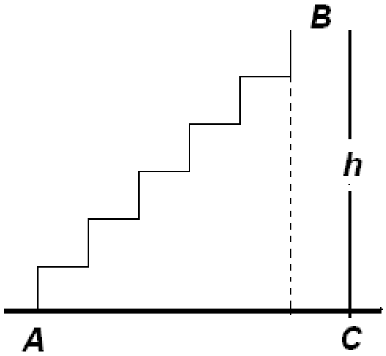
$$W = \frac{m}{2}[v_f^2 - v_i^2]$$

$$W = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

Conservative and non-conservative force

Conservative force

A conservative force may be defined as on for which work done in moving between two points A and B is independent of the path taken between two points. Work done to move particles through stairs is equal to moving particle vertically .The implication of “conservative” in this context is that you could move it from A and B by one path and returns to A by another path with no net loss of energy – any close return path A takes net work zero. Or mechanical energy is conserved



A further implication is that the energy of an object which is subject only to that conservative force is dependent upon its position and not upon the path by which it reached that position. This makes it possible to define a potential energy function which depends upon position only

If a force acting on an object is a function of position only, it is said to be a conservative force and it can be represented by potential energy function which for a one-dimensional case satisfies the derivative condition

$$F(x) = -\frac{dU}{dx}$$

Example for verification

(a) Gravitational potential energy = -mgh

Thus

$$F(h) = -\frac{d(-mgh)}{dh}$$

F(h) = mg

(b) Spring potential energy = (1/2)kx^2

$$F(x) = -\frac{1}{2}k\frac{d(x^2)}{dx}$$

$$F(x) = -kx$$

Non-conservative force

Consider a body moving on a rough surface from A to B and then back from B to A. Work done against frictional forces only add up because in both the displacement work is done against frictional force only. Hence frictional force cannot be considered as a conservative force. It is non-conservative force

Conservation of mechanical energy

Kinetic and potential energy both are forms of mechanical energy. The total mechanical energy of a body or system of bodies will be changed in values if

(a) An external force other than weight causes work to be done(work done by weight is potential energy and is therefore already included in the total mechanical energy)

(b) Some mechanical energy is converted into another form of energy (e.g. sound, heat , light) such a conversion of energy usually takes place when a sudden change in the motion of the system occurs. For instance, when two moving objects collide some mechanical energy is converted into sound energy, which is heard as a bang at the impact.

If neither (a) nor (b) occurs, then the total mechanical energy of a system remains constant.

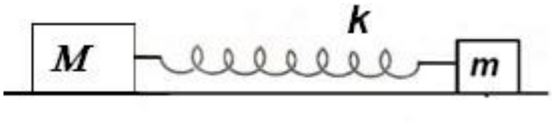
This is the principle of Conservation of Mechanical Energy and can be expressed as

The total mechanical energy of a system remains constant provided that no external work is done and no mechanical energy is converted into another form of energy

When this principle is used in solving problems, a careful appraisal must be made of any external forces, which are acting. Some external forces do work and hence cause a change in the total energy of the system.

Solved Numerical

Q) A spring of force constant k is kept in compressed condition between two blocks of masses m and M on the smooth surface of table as shown in figure. When the spring is released both the blocks move in opposite directions. When the spring attains its original (normal) position, both the blocks lose the contacts with spring. If x is the initial compression of the spring find the speed of block while getting detached from the spring.



Solution

According to law of conservation of energy

Spring Potential energy = Sum of kinetic energy off block

$$\frac{1}{2}kx^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}Mv_2^2$$

From law of conservation of momentum

$mv_1 = Mv_2$

$$v_2 = \frac{m}{M}v_1$$

$$kx^2 = mv_1^2 + M\left(\frac{m}{M}v_1\right)^2$$

$$kx^2 = v_1^2\left(m + \frac{m^2}{M}\right)$$

$$kx^2 = v_1^2\left(\frac{mM + m^2}{M}\right)$$

$$v_1^2 = \frac{kx^2M}{m(M+m)}$$

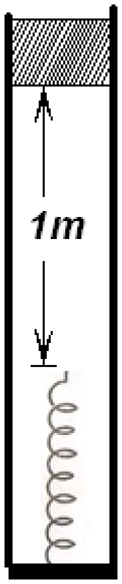
$$v_1 = \sqrt{\frac{kM}{m(M+m)}} \cdot x$$

Similarly

$$v_2 = \sqrt{\frac{km}{M(M+m)}} \cdot x$$

Q) A 20kg body is released from rest, so as to slide in between vertical rails and compresses a spring having a force constant k = 1920 N/m. the spring is 1m below the starting position of the body. The rail offers a resistance of 36N to the motion of the body. Find (i) the velocity of the body just before touching the spring (ii) the distance ,ℓ through which the spring is compressed (iii) the distance 'h' through which the body rebounds up

Solution



(i) Let velocity of the body just before touching the spring be v

Change in k.E = work done

$$\frac{1}{2}mv^2 - 0 = mg \times 1 - 36 \times 1$$

$$\frac{1}{2} \times 20 \times v^2 = 20 \times 9.8 \times 1 - 36 \times 1$$

v = 4 m/s

(ii) Let x be maximum compression of the spring. Then effective height for calculation of potential energy = 1+x

From conservation of energy

Spring potential energy = Change in P.E - Work done against friction

$$\frac{1}{2}kx^2 = mg(1 + x) - 36(1 + x)$$

$$\frac{1}{2} \times 1920 \times x^2 = 20 \times 9.8 \times (1 + x) - 36(1 + x)$$

X = 0.5 m

(iii) Let object bounce up to height h

Potential energy of object = spring potential energy – work against friction

$$mgh = \frac{1}{2}kx^2 - 36h$$

$$20 \times 9.8 \times h = \frac{1}{2} \times 1920 \times (0.5)^2 - 36h$$

$$h = \frac{240}{232} = \frac{30}{29} = 1.03m$$

Q) if work is done on a particle at constant rate, prove that the velocity acquired in describing a distance from rest varies as $x^{1/3}$

Solution

Power is constant, P = F .V = constant (say k)

Now mav = K

$$av = \frac{k}{m}$$

$$v\frac{dv}{dt} = \frac{k}{m}$$

$$v\frac{dv}{dt}\frac{dx}{dx} = \frac{k}{m}$$

$$v^2\frac{dv}{dx} = \frac{k}{m}$$

$$v^2 dv = \frac{k}{m}dx$$

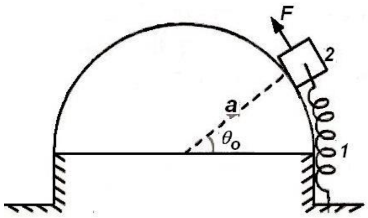
$$\int_0^v v^2 dv = \frac{k}{m}\int_0^x dx$$

$$\frac{v^3}{3} = \frac{k}{m}x$$

$$v^3 \propto x$$

$$v \propto x^{1/3}$$

Q) In the position shown in figure, the spring constant k is undeformed. Find the work done by the variable force F, which is always directed along the tangent to the smooth hemispherical surface on the small block of mass m to shift it from the position 1 to position 2 slowly.

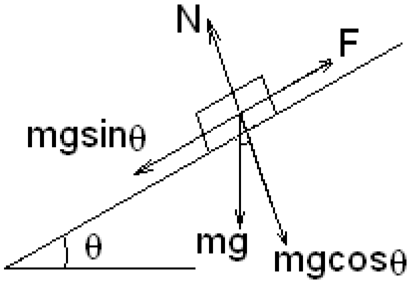


Solution:
From the condition of the equilibrium of the block at any arbitrary angular position $\theta < \theta_o$
F = mgcosθ + kx
Work done in displacing the block through a distance dx = dW
dW = Fdx = (mgcosθ + kx)dx
where x = aθ, dx = adθ
Total work done by the force F on the small block of mass m to shift from the position 1 to position 2 is

$$W = \int F dx = \int_{\theta}^{\theta_0} (\mathrm{mgcos\theta + ka\theta})\mathrm{a d\theta}$$

$$W = mgasin\theta_0 + \frac{ka^2}{2}\theta_0^2$$

Q) A block of mass 2 kg is pulled up on a smooth incline of angle 30° with horizontal. If the block moves with an acceleration of 1 m/s^2, find the power delivered by the pulling force at a time 4 seconds after motion starts. What is the average power delivered during these four seconds after the motion starts?



Solution:

To find power delivered by force at t=4 we have to calculate velocity at t=4 and use formula P = force × velocity

i) Calculation of velocity at t = 4 s

$$V = u + at$$

$$a=1 m/s^2 \ t = 4 \text{ sec given}$$

$$V = 4 m/s$$

ii) Calculation of resultant force

Given resultant acceleration $a = 1 m/s^2, \theta = 30^O$, $g = 9.8 m/s^2$

Thus from the diagram and resolving forces we get equation

$F - mg\sin\theta = ma$

$F = mg\sin\theta + ma$

$F = 2\times9.8\times\sin30 + 2\times1 = 11.8$ N

By substituting values of F and v in equation of power

P = Fv

P = 11.8 × 4 = 47.2 W

To find average power

We have to find total work done by using formula W = FS and then use formula P = W/t

But S is not given , can be calculated using $v^2 = u^2 + 2as$

We have already calculated v = 4m/s, u = 0, a = $1m/s^2$

Thus

$$16 = 0 + 2\times1\times S$$

$$S = 8 \text{ m}$$

Now work done in 4 seconds = Force × displacement

Work done in 4 seconds = 11.8 ×8 = 94.4 J

Average power = Work / time

Average power = 94.4 / 4 = 23.6 W

Q) A block of mass m released from rest onto an ideal non-deformed spring of spring constant k from a negligible height. Neglect the air resistance, find the compression d of the spring.

Solution

(Note: When we attach or put mass on spring, spring under goes motion hence can not solve using formula mg = kx which is the condition for equilibrium)

Block is just kept on spring not allow to fall on spring. Thus Weight of block will press the spring and restoring force will oppose the compression. And equilibrium will be establish.

Let compression be ‘d’ thus potential energy lost by the block = mgd

Potential energy gain by spring = $(/1/2)kd^2$

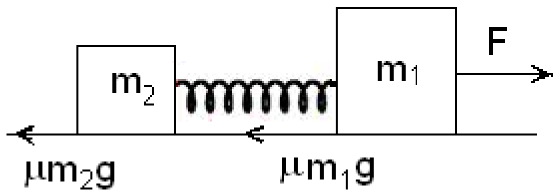
Thus potential energy lost by the block = potential energy gain by spring

$$mgd = (1/2)kd^2$$

$$d = 2mg/k$$

WORK, ENERGY AND POWER

Q) Two masses m_1 and m_2 connected by a non-deformed light spring rest on a horizontal plane. The coefficient of friction between bars and surface is μ. What minimum constant force has to be applied in the horizontal direction to the mass m_1 in order to shift the other mass m_2



Solution:

Note that acceleration of both the masses will be different. Because acceleration of mass m_2 is due to restoring force of spring.

Problem can be solved using law of conservation of energy.

Fist consider mass m_2 do not move and is stationary.

Let x be the displacement of mass m_1 then Work done by force = Fx

This work done is used to overcome friction of mass m_1 and remaining stored as potential energy of spring

Now work done to overcome friction = Frictional force × displacement

$= \mu m_1 g x$

Energy stored in spring since mass m_1 have moved by distance 'x', stretching is spring is 'x' as we have already stated motion of block m_2 is due to restoring force of spring

Thus PE. Of Spring = $(1/2)kx^2$

$$Fx = \mu m_1 g x + (1/2)kx^2 \text{ ------- eq(1)}$$

Since block m_2 moves due to restoring force of spring thus restoring force = frictional force

$$Kx = \mu m_2 g.$$

Substituting value of Kx in equation (1) we get

$$Fx = \mu m_1 g x + (1/2) \mu m_2 g x$$

$$F = \mu g (m_1 + m_2 /2)$$

Q) A block of mass M is attached with a vertical relaxed spring of spring constant k. if the block is released, find maximum elongation in spring.

Solution: Let x be the elongation.

Thus potential energy lost by mass = mgx

Energy gain by spring = $(1/2) kx^2$

From law of conservation of energy

Potential energy loss by mass M = energy gain by spring

$Mgx = (1/2)kx^2$

$X = 2Mg/k$

(Note: When we attach or put mass on spring, spring under goes motion hence can not solve using formula mg = kx which is the condition for equilibrium)

WORK, ENERGY AND POWER

Q) A horse pulls a wagon of 5000 kg from rest against a constant resistance of 90N. the pull exerted initially is 600N and it decreases uniformly with the distance covered to 400N at a distance of 15m from start. Find the velocity of wagon at this point.
Solution:
Force is varying Initially 600 N and goes down to 400 N. Thus average force applied for pull = (600 +400) /2 = 500 N
Resistive force is constant 90N
Effective force = Average force – resistive force
Effective force = 500 – 90 = 410 N
Displacement = 15 m
Thus work done by the force = 410 × 15 = 6150 J
This work by force produces kinetic energy
∴ Kinetic energy of object = work done by force

$$\therefore \frac{1}{2}mv^2 = 6150$$

V= 1.57 m/s

Q) A block of mass 5.0 kg is suspended from the end of a vertical spring, which is stretched by 10cm under the lead of the block. The block is given is given a sharp impulse from below so that it acquires an upward speed of 2.0 m/s. How high will it rise? Take g = 10 m/s^2
Solution:
For equation mg = kx
5 ×10= k ×0.1
K = 500 N
Spring is already elongated by 0.1m thus it already have some potential energy, block attached to spring also have potential energy.
when sharp impulse is given to block gain more potential energy and spring gain potential energy
Thus After sharp impulse given Total energy of system
= previously store potential energy of spring due to 0.1m elongation + given kinetic energy+ potential energy of mass
[Here potential energy of mass is taken positive as below equilibrium point of spring before attaching mass]
When spring gets compressed say by x
= Potential energy of spring + potential energy of block
= $(1/2)kx^2$ - mg(x)
[Here potential energy taken negative as object moved above the equilibrium position of spring before attaching mass]

Thus from law of conservation of energy
$(1/2)\ k(0.1)^2 + (1/2)\ mv^2 + mg(0.1) = (1/2)kx^2 - mg(x)$
$(1/2)500(0.01) + (1/2)5(2)^2 + 5(10)(0.1) = (1/2)(500)x^2 – 5(10)x$
X = 0.1 m and height to which block raise = 0.1 +0.1= 0.2m from equilibrium point before attaching the mass.